

74503/1

BULETINUL INSTITUTULUI POLITEHNIC DIN IAȘI

Tomul XL (XLIV)

Fasc. 1 — 4

Secția I

**MATEMATICĂ
MECANICĂ TEORETICĂ
FIZICĂ**

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The Bulletin of the Polytechnic Institute of Jassy is edited by the
Technical University "Gh. Asachi" (formerly Polytechnic Institute) of Jassy

Secția I

MATEMATICĂ. MECANICĂ TEORETICĂ. FIZICĂ

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ON SPLIT-MODULAIRE EXTENSIONS OF *BT*-ALGEBRAS

BY

ILIE BURDUJAN

0. Introduction. The binary triple algebras (abbreviated *BT*-algebras) were introduced by M. A. Akiyis [1] and K. H. Hofmann and K. Strambach [5] as "tangent algebras" of local analytic loops. They use these algebras as useful tools in investigating properties of analytic loops, so that they do not deal with the algebraic properties of *BT*-algebras.

A systematic study of these algebras has been begun in [2], [3]. These papers are concerned with representations and extensions of *BT*-algebras and with their semi-simplicity and solvability, respectively.

The present paper is concerned with the split-modulaire extensions of *BT*-algebras. For this goal, the so called *quasi-representations* and the *factor sets* will be used.

1. Preliminaries. Throughout this paper, K will denote a commutative ring with unit element.

A *BT*-algebra is a K module A with a bilinear composition $A \times A \rightarrow A$, $(x, y) \rightarrow [x, y]$ and a trilinear composition $A \times A \times A \rightarrow A$, $(x, y, z) \rightarrow \langle x, y, z \rangle$ satisfying the following relations:

$$(A_1) \quad [x, x] = 0, \quad \forall x \in A;$$

$$(A_2) \quad \begin{aligned} &\langle x, y, z \rangle + \langle y, z, x \rangle + \langle z, x, y \rangle - \langle x, z, y \rangle - \langle z, y, x \rangle - \\ &- \langle y, x, z \rangle = [x, [y, z]] + [y, [z, x]] + [z, [x, y]], \quad \forall x, y, z \in A. \end{aligned}$$

An ideal B of a *BT*-algebra A is a K -submodule of A satisfying

$$[A, B] \subset B, \quad \langle A, A, B \rangle \subset B, \quad \langle A, B, A \rangle \subset B, \quad \langle B, A, A \rangle \subset B.$$

For a *BT*-algebra A , the set

$$\mathcal{C}(A) = \{x \in A \mid [x, y] = 0, \quad \langle x, y, z \rangle = \langle y, x, z \rangle = \langle y, z, x \rangle = 0, \quad \forall y, z \in A\}$$

is an ideal of A called the *center* of A .

The definitions of *BT*-morphism, kernel, *BT*-isomorphism, are the usual ones.

Obviously, the kernel of a *BT*-morphism is an ideal of A .

In the following, $L_n(A, B)$ ($n \in \mathbb{N}$) will denote the K -module of all B -valued n -multilinear applications.

For the *BT*-algebra A , let us define (see [2]) the applications

$$\phi: A \rightarrow L_1(A, A), \quad \phi(x)(y) = [x, y],$$

$$\phi_i: A \rightarrow L_2(A, A) \quad (i = 1, 2, 3),$$

$$\phi_1(x)(y, z) = \langle x, y, z \rangle, \quad \phi_2(x)(y, z) = \langle y, x, z \rangle, \quad \phi_3(x)(y, z) = \langle y, z, x \rangle;$$

$$\phi_{ij}: A \times A \rightarrow L_1(A, A), \quad (i, j = 1, 2, 3; i < j),$$

$$\phi_{12}(x, y)(z) = \langle x, y, z \rangle, \quad \phi_{13}(x, y)(z) = \langle x, z, y \rangle, \quad \phi_{23}(x, y) = \langle z, x, y \rangle.$$

The earlier given set $(\phi, \phi_i, \phi_{ij})$ defines the so called *adjoint representation* of A , denoted by Ad_A .

Recall, that an extension

$$(\mathcal{E}) \quad 0 \longrightarrow A \xrightarrow{\rho} E \xrightarrow{\sigma} B \longrightarrow 0,$$

of *BT*-algebra B by the *BT*-algebra A is called *split-modulaire extension* if there exists a section $\theta: B \rightarrow E$ (i.e. a K -linear mapping satisfying: $\sigma \cdot \theta = \text{id}_B$).

2. Quasi-Representations of *BT*-algebras. Let A, B be two *BT*-algebras over K . The K -linear applications

$$(1) \quad \begin{aligned} \phi: B \rightarrow L_1(A, A), \quad \phi_i: A \rightarrow L_2(A, A) \\ (i = 1, 2, 3), \end{aligned}$$

and the K -bilinear applications

$$(2) \quad \phi_{ij}: B \times B \rightarrow L_1(A, A), \quad (i, j = 1, 2, 3; i < j), \quad \psi: B \times B \rightarrow A,$$

are given. Then the following K -linear applications

$$(3) \quad \phi^i: B \rightarrow L_2(A, A) \quad (i = 1, 2, 3); \quad \Phi: B \rightarrow L_2(A, A),$$

are constructed by

$$(4) \quad \phi_x^1(y, z) = \phi_x([y, z]_A), \quad \phi_x^2(y, z) = -[y, \phi_x z]_A, \quad \phi_x^3(y, z) = -[\phi_x y, z]_A,$$

for $\forall x \in B$ and $\forall y, z \in A$,

$$(5) \quad \Phi = \phi_1 - \phi_2 + \phi_3 - \frac{1}{2}(\phi^1 + \phi^2 + \phi^3)$$

(here it has been used the notations: $\phi_x(y) = \phi(x)(y)$, $\phi_x^i(y, z) = \phi^i(x)(y, z)$, $\forall x \in B$, $\forall y, z \in A$; these notations will be used in the following, too). Also, let us define the composition

$$(6) \quad \phi_x * \phi_y = \phi_x \cdot \phi_y - \phi_{12} + \phi_{13} - \phi_{23}, \quad \forall x, y \in B.$$

Definition 1. A *quasi-representation* of *BT*-algebra B in A is a set of seven K -multilinear applications $(\phi, \phi_i, \phi_{ij}, \psi)$ ($i, j = 1, 2, 3$; $i < j$) as before, satisfying

$$\Phi(x) = \Phi(x) \cdot s, \quad \forall x \in B,$$

$$(7) \quad [\phi_x, \phi_y]^*_{\text{def}} = \phi_x * \phi_y - \phi_y * \phi_x = \phi_{[x, y]} + \text{ad}_{\psi(x, y)}^A, \quad \forall x, y \in B,$$

$$\psi = -\psi \cdot s,$$

where $s(x, y) = (y, x)$, and $\text{ad}_x^A y = [x, y]_A$, $\forall x, y \in A$.

The set of all quasi-representations of B in A will be denoted by $\mathcal{QR}(B, A)$.

Remark 1. Any representation $R \in \mathcal{R}(B, a)$ (see [2], Def. 6) is a quasi-representation having $\psi = 0$.

Remark 2. The quasi-representations of a *BT*-algebra B in a K -modul A (viewed as a trivial *BT*-algebra) is a representation.

3. Split-modulaire Extensions. Let

$$(E) \quad 0 \longrightarrow A \xrightarrow{\rho} E \xrightarrow{\sigma} B \longrightarrow 0,$$

be a split-modulaire extension, having the section θ . Then, the adjoint representation Ad_E allows us to consider the following mappings of type (1) and

(2):

$$\phi_b(a) = \rho^{-1}([\theta(b), \rho(a)]_E),$$

$$\phi_1(b)(a, a') = \rho^{-1}(\langle \theta(b), \rho(a), \rho(a') \rangle_E),$$

$$\phi_2(b)(a, a') = \rho^{-1}(\langle \rho(a), \theta(b), \rho(a') \rangle_E),$$

$$\phi_3(b)(a, a') = \rho^{-1}(\langle \rho(a), \rho(a'), \theta(b) \rangle_E);$$

(8)

$$\phi_{12}(b, b')(a) = \rho^{-1}(\langle \theta(b), \theta(b'), \rho(a) \rangle_E),$$

$$\phi_{13}(b, b')(a) = \rho^{-1}(\langle \theta(b), \rho(a), \theta(b') \rangle_E),$$

$$\phi_{23}(b, b')(a) = \rho^{-1}(\langle \rho(a), \theta(b), \theta(b') \rangle_E),$$

$$\psi(b, b') = \rho^{-1}([\theta(b), \theta(b')]_E - \theta([b, b']_B)), \quad \forall b, b' \in B, \forall a, a' \in A.$$

The corresponding mappings of type (4) are

$$\phi^1(b)(a, a') = \rho^{-1}([\theta(b), [\rho(a), \rho(a')]_E]_E),$$

$$\phi^2(b)(a, a') = -\rho^{-1}([\rho(a), [\theta(b), \rho(a')]_E]_E),$$

$$\phi^3(b)(a, a') = -\rho^{-1}([\theta(b), \rho(a)]_E, \rho(a')]_E).$$

By direct computations, via the axioms of *BT*-algebras, it may be proved the following result.

Proposition 1. *The set of mappings $(\phi, \phi_i, \phi_{ij}, \psi)$ ($i, j=1, 2, 3$; $i < j$) defined by (8) gives a quasi-representation of B in A (induced by Ad_A and θ).*

4. Equivalent Split-modulaire Extensions. Definition 2. Two split-modulaire extensions

$$(\mathcal{E}_1) \quad 0 \longrightarrow A \xrightarrow{\rho_1} E_1 \xrightleftharpoons[\theta_1]{\sigma_1} B \longrightarrow 0,$$

$$(\mathcal{E}_2) \quad 0 \longrightarrow A \xrightarrow{\rho_2} E_2 \xrightleftharpoons[\theta_2]{\sigma_2} B \longrightarrow 0,$$

are called *equivalent* if there exists a BT-morphism $\tau: E_1 \rightarrow E_2$ such that the following diagram is a commutative one

$$(9) \quad \begin{array}{ccccc} & & E_1 & & \\ & \nearrow \rho_1 & \downarrow \tau & \nwarrow \sigma_1 & \\ 0 & \longrightarrow & A & \longrightarrow & B \longrightarrow 0 \\ & \searrow \rho_2 & \downarrow \tau & \nwarrow \sigma_2 & \\ & & E_2 & & \end{array}$$

θ_1 (between E_1 and B), θ_2 (between E_2 and B)

It is to remark that if two extensions are equivalent and one of them is split-modulaire, then the other one is a split-modulaire extension, too.

It may be proved the

Proposition 2. For two equivalent split-modulaire extensions $(\mathcal{E}_1)$ and $(\mathcal{E}_2)$ (see Definition 2) the corresponding quasi-representations R_1 and R_2 (via formulas (8)) are coincident.

5. The Factor Set of a Split-modulaire Extension. Let

$$(8) \quad 0 \longrightarrow A \xrightarrow{\rho} E \xrightleftharpoons[\theta]{\sigma} B \longrightarrow 0,$$

be a split-modulaire extension. Then, the functions

$$(10) \quad \begin{aligned} \psi_1: B \times B \rightarrow E, \quad \psi_1(b, b') &= [\theta(b), \theta(b')]_E - \theta([b, b']_B), \\ \kappa_1: B \times B \times B \rightarrow E, \\ \kappa_1(b, b', b'') &= \langle \theta(b), \theta(b'), \theta(b'') \rangle_E - \theta(\langle b, b', b'' \rangle_B), \quad \forall b, b' \in B, \end{aligned}$$

have the properties: $\psi_1(B \times B) \subset \rho(A)$, $\kappa_1(B \times B \times B) \subset \rho(A)$, that enables us to define the functions $\psi: B \times B \rightarrow A$ and $\kappa: B \times B \times B \rightarrow A$ by

$$(11) \quad \psi = \rho^{-1} \cdot \psi_1, \quad \kappa = \rho^{-1} \cdot \kappa_1.$$

Definition 4. The mapping κ is called the factor set of extension $(\mathcal{E})$.

By direct computation one find

$$(12) \quad \psi(b, b') = -\psi(b', b),$$

$$\begin{aligned}
 & \kappa(b, b', b'') + \kappa(b', b'', b) + \kappa(b'', b, b') - \kappa(b, b'', b') - \kappa(b'', b', b) - \\
 (13) \quad & - \kappa(b', b, b'') = \psi(b, [b', b'']) + \psi(b', [b'', b]) + \psi(b'', [b, b']) + \\
 & + \phi_b(\psi(b', b'')) + \phi_b/(\psi(b'', b)) + \phi_b/(\psi(b, b')), \quad \forall b, b', b'' \in B.
 \end{aligned}$$

Thus, we have

Proposition 3. *If there exists an extension ($\mathcal{E}$) of B by A , then there exists a set of nine functions $(\phi, \phi_i, \phi_{ij}, \psi, \kappa)(i, j=1, 2, 3; i < j)$ satisfying (7) and (13).*

Also, it may be proved the

Proposition 4. *Two equivalent split-modulaire extensions lead to the same set $(\phi, \phi_i, \phi_{ij}, \psi, \kappa)(i, j=1, 2, 3; i < j)$ (i.e. to the same pair (R, κ) , where $R=(\phi, \phi_i, \phi_{ij}, \psi)$ is the quasi-representation induced by extension).*

6. A Semidirect Product of BT-algebras. Let A, B be two BT-algebras and $(\phi, \phi_i, \phi_{ij}, \psi, \kappa)(i, j=1, 2, 3; i < j)$ a set of nine functions satisfying equations (7) and (13).

We define on $A \times B$ the compositions

$$(14) \quad [(a, b), (a', b')] = [a, a'] + \phi_b(a') - \phi_b/(a) + \psi(b, b'), [b, b']];$$

$$\begin{aligned}
 & \langle (a, b), (a', b'), (a'', b'') \rangle = (\langle a, a', a'' \rangle + \phi_1(b)(a', a'') + \phi_2(b'/(a, a'')) + \\
 (15) \quad & + \phi_3(b'/(a, a')) + \phi_{12}(b, b')(a'') + \phi_{13}(b, b'/(a')) + \phi_{23}(b', b'/(a)) + \\
 & + \kappa(b, b', b''), \langle b, b', b'' \rangle), \quad \forall (a, b), (a', b'), (a'', b'') \in A \times B.
 \end{aligned}$$

Proposition 5. *The K -multilinear compositions (14), (15) organize $A \times B$ as a BT-algebra.*

Proof. Since the two compositions are K -multilinear it is sufficient to prove the achievement of the two axioms (A_1) and (A_2) only in the following four cases: I. $(a, 0), (a', 0), (a'', 0)$; II. $(a, 0), (a', 0), (0, b'')$; III. $(a, 0), (0, b'), (0, b'')$; IV. $(0, b), (0, b'), (0, b'')$, that is just a computation problem.

Definition 5. The BT-algebra on $A \times B$ defined by compositions (14), (15) is called the *semidirect product* of BT-algebras A and B corresponding to pair (R, κ) , where $R \in \mathcal{QR}(B, A)$ is $R=(\phi, \phi_i, \phi_{ij}, \psi)(i, j=1, 2, 3; i < j)$. It is denoted by $A \times_{(R, \kappa)} B$.

Proposition 6. Let A, B be two BT-algebras, $R=(\phi, \phi_i, \phi_{ij}, \psi)(i, j=1, 2, 3; i < j)$ a quasi-representation and κ a factor set such that (7) and (13) are fulfilled. Then, there exists a split-modulaire extension $(\mathcal{E}_0)$ with the section θ_0 of B by A such that R is just the quasi-representation of B in A induced by $(\mathcal{E}_0)$, while κ is the factor set corresponding to θ_0 .

P r o o f. Let consider the split-modulaire extension

$$(\mathcal{E}_0) \quad 0 \longrightarrow A \xrightarrow{\rho_0} E_0 = A \times_{(R, \kappa)} B \xrightleftharpoons[\theta_0]{\sigma_0} B \longrightarrow 0,$$

where $\rho_0(a) = (a, 0)$, $\sigma_0(a, b) = b$ and $\theta_0(b) = (a, b)$. Also, let consider $(\bar{\phi}, \bar{\phi}_i, \bar{\phi}_{ij}, \bar{\psi}, \bar{\kappa}) = (\bar{R}, \bar{\kappa})$, $(i, j=1, 2, 3; i < j)$ the set of functions associated with this split-modulaire extension. By direct computations it results $R = \bar{R}$ and $\kappa = \bar{\kappa}$.

Moreover, it may be proved

Proposition 7. Let

$$(\mathcal{E}) \quad 0 \longrightarrow A \xrightarrow{\rho} E_1 \xrightleftharpoons[\theta]{\sigma} B \longrightarrow 0,$$

be a split-modulaire extension, inducing the quasi-representation $R=(\phi, \phi_i, \phi_{ij}, \psi)$ and the factor set κ . Moreover, let us consider the split-modulaire extension

$$(\mathcal{E}_0) \quad 0 \longrightarrow A \xrightarrow{\rho_0} E_0 = A \times_{(R, \kappa)} B \xrightleftharpoons[\theta_0]{\sigma_0} B \longrightarrow 0.$$

Then, the mapping $\tau: E_0 \rightarrow E$, $\tau(a, b) = \rho(a) + \theta(b)$ is an equivalence between the two extensions $(\mathcal{E})$ and $(\mathcal{E}_0)$.

So, we obtain the following main result

Theorem 1. The set of all equivalence classes of split-modulaire extensions of BT-algebra B by the BT-algebra A are in 1-1-correspondence with the set of all sets of functions $(\phi, \phi_i, \phi_{ij}, \psi, \kappa)(i, j=1, 2, 3; i < j)$ satisfying (7) and (13).

7. A Particular Case. For some split-modulaire extensions it is possible that the section defining the split-modulaire characteristic to be a morphism for ternary compositions, only. Then $\kappa=0$ and all results previously proved may be stated in a corresponding form. For example we state

Theorem 1'. The set of all equivalence classes of split-modulaire

extensions having its defining sections as morphisms of ternary compositions are in 1-1-correspondence with the set $QR(B, A)$.

Received December 20, 1993

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ASUPRA EXTENSIUNILOR SPLIT-MODULARE ALE BT-ALGEBRELOR

(Rezumat)

Prezenta lucrare este o continuare a lui [2] și dă un rezultat de clasificare a extensiunilor split-modulare, folosind noțiunea de cvasi-reprezentare.

THE TRANSVERSAL VECTOR BUNDLE OF A LIGHTLIKE SUBMANIFOLD

BY

AUREL BEJANCU

Let $(\bar{M}, \bar{g})$ be a real $(m+n)$ -dimensional semi-Riemannian manifold, where $m > 1$, $n > 1$ and $\bar{g}$ is a semi-Riemannian metric on $\bar{M}$ of constant index $q \in \{1, \dots, m+n-1\}$. Hence $(\bar{M}, \bar{g})$ is never a Riemannian manifold.

Suppose M is a submanifold of $\bar{M}$ of codimension n . The conditions $m > 1$ and $n > 1$ imply that M is neither a curve nor a hypersurface of $\bar{M}$. Denote by g the induced tensor on M by $\bar{g}$, i.e., we have $g_u(X_u, Y_u) = \bar{g}_u(X_u, Y_u)$, for any $u \in M$ and $X_u, Y_u \in T_u M$. There exist three possibilities:

1. g is nondegenerate on TM ;
2. g is degenerate on TM ;
3. g is nondegenerate on $TM|_A$ and degenerate on $TM|_B$ where $A \cup B = M$.

In the first case, the study of the immersion of M in $\bar{M}$ has many similarities with that one of Riemannian submanifolds (cf. Chen [3], Dajczer [4] and O'Neill [9]). The second class of submanifolds was studied recently by a few people (Duggal — Bejancu [5], Küpeli [6], [7]) and is going to be the matter of study for the present paper. The third class falls in the category of singular semi-Riemannian geometry (cf. Larsen [8]).

As we have seen, in case of lightlike curves (cf. Bejancu [1]) and lightlike hypersurfaces (cf. Bejancu — Duggal [2]), the tangent bundle and the normal bundle of such submanifolds have a non-void intersection. The same situation we meet in case of lightlike submanifolds of dimension and codimension greater than 1. That is why, the purpose of the present paper is to construct the transversal vector bundle of a lightlike submanifold M of $\bar{M}$ and as a consequence obtain the local field of frames on M along M .

In order to clarify the concept we are dealing with, take $u \in M$ and consider

$$T_u M^\perp = \{V_u \in T_u \bar{M}; \bar{g}_u(V_u, W_u) = 0, \quad \forall W_u \in T_u M\}.$$

In case the induced tensor g is nondegenerate on TM , at each point $u \in M$ both $T_u M$ and $T_u M^\perp$ are nondegenerate. Besides, $T_u M$ and $T_u M^\perp$ are complementary orthogonal vector subspaces of $T_u \bar{M}$. Otherwise, both $T_u M$ and $T_u M^\perp$ are degenerate orthogonal subspaces but not anymore complementary subspaces. As the dimension of $\text{Rad } T_u M = T_u M \cap T_u M^\perp$ depends on the point $u \in M$, we give the following definition:

The submanifold M of $\bar{M}$ is said to be *r-lightlike* (*r-degenerate*, *r-isotropic*) submanifold if the mapping

$$\text{Rad } TM: u \in M \rightarrow \text{Rad } T_u M,$$

defines a smooth distribution on M of rank $r > 0$. Then we call $\text{Rad } TM$ the *radical* (*lightlike*, *null*, *isotropic*) *distribution* on M . Also we say that g is a *r-lightlike* (*r-degenerate*) *metric* on M .

It is important to note that the condition from the definition of a lightlike submanifold has a local character. More precisely, it is easy to check the following result.

Theorem 1. Let M be a submanifold of $(\bar{M}, \bar{g})$. Then the following assertions are equivalent:

- i) M is a *r-lightlike* submanifold;
- ii) On each coordinate neighborhood $U \subset M$ the mapping

$$\text{Rad } TU: u \in U \rightarrow \text{Rad } T_u U,$$

defines on U a smooth distribution of rank $r > 0$;

- iii) On each coordinate neighborhood $U \subset M$ the induced tensor g by $\bar{g}$ has a constant rank $m - r$, $0 < r \leq m$.

In order to construct a transversal vector bundle of a lightlike submanifold M of $\bar{M}$ we examine four cases with respect to the dimension and codimension of M and rank of $\text{Rad } TM$.

Case I ($0 < r < \min \{m, n\}$). Consider a complementary distribution SM to $\text{Rad } TM$ in TM and suppose it is of constant index on M . Then SM is orthogonal to $\text{Rad } TM$ and nondegenerate with respect to g . Hence we have the orthogonal direct sum

$$(1) \quad TM = \text{Rad } TM \perp SM.$$

Next, as in case of the theory of nondegenerate submanifolds consider the vector bundle

$$TM^\perp = \bigcup_{u \in M} T_u M^\perp.$$

But contrary to the nondegenerate case, $TM^\perp$ is not complementary to TM in TM since $\text{Rad } TM = TM \cap TM^\perp$ is now a distribution on M of rank $r > 0$. Further on, consider a complementary vector bundle sM to $\text{Rad } TM$ in $TM^\perp$.

It follows that sM is nondegenerate too with respect to g and $TM^\perp$ has the orthogonal direct decomposition

$$(2) \quad TM^\perp = \text{Rad } TM \perp sM.$$

We call SM and sM a *screen distribution* and a *screen transversal vector bundle* of the r -lightlike submanifold M .

Taking into account that SM is a nondegenerate vector subbundle of $\bar{TM}$ over M , we put

$$(3) \quad \bar{TM}|_M = SM \perp SM^\perp,$$

where $SM^\perp$ is the complementary orthogonal vector bundle of SM in $\bar{TM}|_M$. Note that sM is a vector subbundle of $SM^\perp$ and since both sM and $SM^\perp$ are nondegenerate we obtain the orthogonal direct decomposition

$$(4) \quad SM^\perp = sM \perp sM^\perp.$$

It is easy to see that $sM^\perp$ is of rank $2r$.

As theory we develop is mainly based on both SM and sM we frequently denote a r -lightlike submanifold by (M, g, SM, sM) . The following range for indices is used in this section: $i, j, k, \dots \in \{1, \dots, r\}$; $a, b, c, \dots \in \{r+1, \dots, m\}$; $\alpha, \beta, \gamma, \dots \in \{r+1, \dots, n\}$. We make use of the Einstein convention, that is, repeated indices with one upper index and one lower index denote summation over their range.

L e m m a 1. *Let (M, g, SM, sM) be a 1-lightlike submanifold of $(\bar{M}, \bar{g})$. Suppose $\mathcal{U}$ is a coordinate neighborhood of M and ξ is a nonzero section of $\text{Rad } \bar{TM}|_{\mathcal{U}}$. Then there exists a unique section N of $sM|_{\mathcal{U}}$ such that*

$$(5) \quad \bar{g}(N, \xi) = 1$$

and

$$(6) \quad \bar{g}(N, N) = 0.$$

P r o o f. As $sM^\perp$ is nondegenerate and $\text{Rad } \bar{TM}$ is a vector subbundle of $sM^\perp$, there exists a section V of $sM|_{\mathcal{U}}$ such that $\bar{g}(V, \xi) \neq 0$ on $\mathcal{U}$. Thus any section N of $sM|_{\mathcal{U}}$ is expressed $N = \alpha V + \beta \xi$, since $\text{rank } sM^\perp = 2$. Then by direct calculations it follows that N satisfies (5) and (6) iff

$$(7) \quad N = \frac{1}{\bar{g}(V, \xi)} \left\{ V - \frac{\bar{g}(V, V)}{2\bar{g}(V, \xi)} \xi \right\}.$$

Finally, it is easy to check that N does not depend on V .

Throughout the paper denote by $\Gamma(E)$ the module of smooth sections of a vector bundle E over M .

L e m m a 2. *Let (M, g, SM, sM) be a r -lightlike submanifold of $(\bar{M}, \bar{g})$ with $r > 1$. Suppose $\mathcal{U}$ is a coordinate neighborhood of M and*

$\{\xi_i\}, i \in \{1, \dots, r\}$ is a basis of $\Gamma(\text{Rad } TM|_U)$. Then there exist smooth sections $\{N_i\}, i \in \{1, \dots, r\}$ of $sM^\perp$ such that

$$(8) \quad g(\bar{N}_i, \xi_j) = \delta_{ij}$$

and

$$(9) \quad \bar{g}(\bar{N}_i, \bar{N}_j) = 0,$$

for any $i, j \in \{1, \dots, r\}$.

P r o o f. Consider a complementary vector bundle F of $\text{Rad } TM$ in $sM^\perp$ and choose a basis $\{V_i\}, i \in \{1, \dots, r\}$ of $\Gamma(F|_U)$. Thus the sections we are looking for are expressed as follows

$$(10) \quad N_i = \sum_{k=1}^r \{A_{ik} \xi_k + B_{ik} V_k\},$$

where A_{ij} and B_{ij} are smooth functions on U . Then $\{N_i\}$ satisfy (8) iff

$$\sum_{k=1}^r \{B_{ik} \bar{g}_{ik}\} = \delta_{ij},$$

where $\bar{g}_{jk} = \bar{g}(\xi_j, V_k)$, $k, j \in \{1, \dots, r\}$. Note that $G = \det[\bar{g}_{kj}]$ vanishes nowhere on U , otherwise $sM^\perp$ is degenerate. It follows that the above system has the unique solution

$$(11) \quad B_{ik} = \frac{(\bar{g}_{ik})'}{G},$$

where $(\bar{g}_{ik})'$ are the algebraic complements of the elements $\bar{g}_{ik}$ in G . Finally, we obtain that (9) is equivalent with

$$(12) \quad A_{ij} + A_{ji} + \sum_{k,h=1}^r \{B_{ik} B_{jh} \bar{g}(V_k, V_h)\} = 0,$$

which proves the existence of A_{ij} .

By using (8) and (9) it is easy to check that $B = \{\xi_1, \dots, \xi_r, N_1, \dots, N_r\}$ is a basis of $\Gamma(sM|_U)$. On the other hand, from (12) we see that, in general, the set of local sections $\{N_i\}, i \in \{1, \dots, r\}$ is not unique even in case we use the same vector bundle F .

The main difference between the theory of lightlike submanifolds and the classical theory of Riemannian (semi-Riemannian) submanifolds is caused by the fact that in the first case a part of $TM^\perp$ (the radical distribution) lies in the tangent bundle of the submanifold. Thus a crucial problem in this theory is to replace that part by a vector subbundle of $TM|_M$ whose sections are nowhere tangent to M . This is going to be done in theorems which follow in this paper.

T h e o r e m 1. Let (M, g, SM, sM) be a 1-lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ and U be a coordinate neighborhood of M . Then there exists a unique vector subbundle tM of $sM^\perp$ of rank 1 and for any

$\xi \in \Gamma(\text{Rad } TM|_U)$, $\xi \neq 0$ on U , there exists a unique $N \in \Gamma(sM|_U^\perp)$ satisfying (5) and (6).

P r o o f. Apply Lemma 1 and on each $U \subset M$ obtain a unique section N of $sM^\perp$ that satisfies (5) and (6). Consider another neighborhood of coordinates $U^* \subset M$ such that $U \cap U^* \neq \emptyset$ and take $\xi^* \in \Gamma(\text{Rad } TM|_{U^*})$, $\xi^* \neq 0$.

Then apply Lemma 1 on U^* and obtain $N^* \in \Gamma(sM|_{U^*}^\perp)$ satisfying (5) and (6) but with respect to ξ^* . As $\xi^* = \alpha\xi$ and $V^* = \beta V + \gamma\xi$ on $U \cap U^*$, where α, β, γ are smooth functions on $U \cap U^*$, by direct calculations obtain $N^* = (1/\alpha)N$. Hence we obtain the vector bundle tM whose local sections N are given by (7). Finally, the uniqueness of N on U implies the uniqueness of tM .

C o r o l l a r y 1. Let (M, g, SM, sM) be a r -lightlike submanifold of a Lorentz manifold $(\bar{M}, \bar{g})$. Then $r=1$ and there exists a vector bundle tM satisfying conditions from Theorem 1.

P r o o f. Since M is Lorentz it is of index $q=1$. As $0 < r \leq q$ we get $r=1$ and then apply Theorem 1.

T h e o r e m 2. Let (M, g, SM, sM) be a r -lightlike submanifold, $r > 1$, of a semi-Riemannian manifold $(M, \bar{g})$. Then there exists a complementary vector bundle tM of $\text{Rad } TM$ in $sM^\perp$ such that $\{N_i\}$, $i \in \{1, \dots, r\}$ from Lemma 2 be a basis of $\Gamma(tM|_U)$.

P r o o f. Consider the linear independent local sections $\{N_i\}$ from Lemma 2 but in case we choose $A_{ij} = A_{ji}$. Hence from (12) we obtain

$$(13) \quad A_{ij} = -\frac{1}{2} \sum_{k,h=1}^r \{B_{ik} B_{jh} \bar{g}(V_k, V_h)\},$$

where B_{ik} are uniquely determined by (11). Hence on each coordinate neighborhood $U^* \subset M$ we have N_i from (10) with A_{ij} and B_{ij} given by (13) and (11) respectively. Next, consider another coordinate neighborhood $U \subset M$ such that $U \cap U^* \neq \emptyset$. Then choose $\{\xi_i^*\}$ and $\{V_i^*\}$ as basis of $\Gamma(\text{Rad } TM|_{U^*})$ and $\Gamma(F|_{U^*})$ respectively and obtain the local vector fields

$$(14) \quad N_i^* = \sum_{j=1}^r \{A_{ij}^* \xi_j^* + B_{ij}^* V_j^*\},$$

where

$$(15) \quad B_{ij}^* = \frac{(\bar{g}_{ij}^*)'}{G^*}, \quad \bar{g}_{ij}^* = \bar{g}(\xi_i^*, V_j^*), \quad G^* = \det[\bar{g}_{ij}^*],$$

and

$$(16) \quad A_{ij}^* = -\frac{1}{2} \sum_{h,k=1}^r \left\{ B_{ik}^* B_{jh}^* \bar{g}(V_k^*, V_h^*) \right\}.$$

On $H \cap H^*$ we put

$$(17) \quad \xi_i^* = C_i^j \xi_j, \quad V_i^* = E_i^j V_j,$$

where C_i^j and E_i^j are smooth functions on $H \cap H^*$. Denote by C and E the determinants of invertible matrices $[C_i^j]$ and $[E_i^j]$ respectively. Next we note the algebraic identities

$$(18) \quad G^* = GCE, \quad (\bar{g}_{ij}^*)' = (\bar{g}_{hk})' (C_i^h)' (E_j^k)'. \quad \bar{g}_{ij}^* = \bar{g}_{hk} C_i^h E_j^k.$$

Then by using (15), (17), (18), (11) and usual properties of algebraic complements obtain

$$(19) \quad \begin{aligned} \sum_{j=1}^r \{B_{ij}^* V_j^*\} &= \frac{1}{G^*} \sum_{j=1}^r \left\{ (\bar{g}_{ij}^*)' E_j^t \right\} V_t = \frac{1}{GCE_{j=1}^r} \left\{ (E_j^k)' (E_j^t)' \right\} (\bar{g}_{hk})' E_j^t V_t = \\ &= \frac{1}{C} (C_i^h)' \sum_{k=1}^r \left\{ \frac{(\bar{g}_{hk})'}{G} V_k \right\} = \frac{1}{C} (C_i^h)' \sum_{k=1}^r \{B_{hk} V_k\}. \end{aligned}$$

On the other hand, by using (16), (17), (19) and (13) we infer

$$(20) \quad \begin{aligned} \sum_{j=1}^r \{A_{ij}^* \xi_j^*\} &= -\frac{1}{2} \sum_{j=1}^r \left\{ \bar{g} \left[\sum_{k=1}^r B_{ik}^* V_k^*, \sum_{h=1}^r B_{jh}^* V_h^* \right] \xi_j^* \right\} = \\ &= -\frac{1}{2C^2} \sum_{j=1}^r \left\{ \bar{g} \left[\sum_{k=1}^r B_{sk} (C_i^s)' V_k, \sum_{h=1}^r B_{ph} (C_j^p)' V_h \right] C_j^t \right\} \xi_t = \\ &= -\frac{1}{2C} \sum_{t,k,h=1}^r \left\{ B_{sk} B_{th} g(V_k, V_h) (C_i^s)' \xi_t \right\} = \frac{1}{C} \sum_{t=1}^r \left\{ A_{st} (C_i^s)' \xi_t \right\}. \end{aligned}$$

Taking account of (19) (20) and (10) in (14) obtain

$$(21) \quad N_i^* = \frac{1}{C} (C_i^h)' \sum_{k=1}^r \{B_{hk} V_k + A_{hk} \xi_k\} = \frac{1}{C} (C_i^h)' N_h.$$

Hence there exists a vector bundle tM of rank r spanned locally on each H by N_i given by (10), (11) and (13). Moreover, tM is complementary to $\text{Rad } TM$ in $sM^\perp$. In fact, suppose there exists a point $u \in M$ and a vector $W_u \in \Gamma(\text{Rad } T_u M) \cap \cap (tM)_u$. Thus $W_u = W^i(u) N_i(u) = W^i(u) \xi_i(u)$, which together with (8) imply

$$\bar{g}_u(W_u, \xi_j(u)) = \bar{g}_u(W^i(u) N_i(u), \xi_j(u)) = W^j(u),$$

and

$$\bar{g}_u(W_u, \xi_j(u)) = \bar{g}_u(W^i(u) \xi_i(u), \xi_j(u)) = 0.$$

Hence $W_u = 0$, and this completes the proof of the theorem.

We call tM constructed in Theorem 2 the *canonical lightlike transversal vector bundle* of M with respect to the pair (SM, sM) . Any other complementary vector bundle of $\text{Rad } TM$ in $sM^\perp$ satisfying conditions as tM from Theorem 2 is called a *lightlike transversal vector bundle* of M .

Next, consider the vector bundle

$$(22) \quad NM = tM \perp sM,$$

where tM is either the canonical lightlike transversal vector bundle or any other lightlike transversal vector bundle. Clearly, NM is of rank n and $(NM)_u \cap T_u M = \{0_u\}$ for any $u \in M$. Thus NM is a complementary (but not orthogonal) vector bundle to TM in $\bar{TM}$. That why we call NM the *canonical transversal vector bundle* of M or a *transversal vector bundle*, according as tM is canonical or not, respectively.

By using (1) and (22) we obtain

$$(23) \quad \bar{TM}|_M = TM \oplus NM = SM \perp sM \perp (tM \oplus \text{Rad } TM).$$

Hence we obtain a local quasi-orthogonal field of frames on $\bar{M}$ along M

$$(24) \quad \{\xi_i, X_\alpha, N_i, W_\alpha\},$$

where $\{\xi_i\}$ and $\{N_i\}$ are lightlike basis of $\Gamma(\text{Rad } TM|_M)$ and $\Gamma(tM|_M)$ respectively, related by (8) and $\{X_\alpha\}$ and $\{W_\alpha\}$ are orthonormal basis of $\Gamma(SM|_M)$ and $\Gamma(sM|_M)$ respectively. As each pair $\{\xi_i, N_i\}, i \in \{1, \dots, r\}$, is a hyperbolic plane we conclude that $\text{Rad } TM \oplus tM$ is nondegenerate of constant index r on M . Hence by a result in O'Neill ([9], p. 51) from (23) obtain

$$(25) \quad q = r + \text{index}(SM)_u + \text{index}(sM)_u, \quad \forall u \in M.$$

As SM is supposed to be of constant index it follows sM is of constant index too. Moreover, from (25) it follows

Proposition 1. *Let M be a lightlike submanifold of a Lorentz manifold $(M, \bar{g})$. Then any screen distribution and any screen vector bundle of M is a Riemannian vector bundle.*

Case II ($1 < r = n < m$). In this case $\text{Rad } TM = TM^\perp$, and thus the former normal bundle from the classical theory of Riemannian submanifolds becomes a distribution on the lightlike submanifold. Consider as in Case I the screen distribution SM on M and obtain

$$(26) \quad TM = SM \perp TM^\perp.$$

As $sM = \{0\}$ we need to replace Lemma 2 by

L e m m a 3. Let (M, g, SM) be an n -lightlike submanifold of $(\bar{M}, \bar{g})$. Suppose U is a coordinate neighborhood of M and $\{\xi_1, \dots, \xi_n\}$ is a basis of $\Gamma(TM|_U)^\perp$. Then there exist smooth sections $\{N_1, \dots, N_n\}$ of $TM|_U$ such that

$$(27) \quad \bar{g}(N_i, \xi_j) = \delta_{ij},$$

and

$$(28) \quad \bar{g}(N_i, N_j) = 0, \quad \bar{g}(N_i, X) = 0,$$

for any $i, j \in \{1, \dots, n\}$ and $X \in \Gamma(SM|_U)$.

P r o o f. As both SM and $TM|_M$ are nondegenerate we set

$$(29) \quad T\bar{M}|_M = SM \perp SM^\perp.$$

Note that rank of $SM^\perp$ is $2n$ and $TM^\perp$ is a vector subbundle of $SM^\perp$. Consider a complementary vector bundle F of $TM^\perp$ in $SM^\perp$. Define N_i by the same formula (10) and from this point the proof follows as in Lemma 2.

Moreover, the proof of the following theorem runs as one of Theorem 2.

T h e o r e m 3. Let (M, g, SM) be an n -lightlike submanifold of $(\bar{M}, \bar{g})$. Then there exists a complementary vector bundle TM of $TM^\perp$ in $SM^\perp$ such that $\{N_1, \dots, N_m\}$ from Lemma 3 be a basis of $\Gamma(TM|_U)$.

In this case (23) becomes

$$(30) \quad T\bar{M}|_M = TM \oplus TM = SM \perp (TM \oplus TM^\perp).$$

Therefore the local quasi-orthogonal field of frames on $\bar{M}$ along M is given by

$$(31) \quad \{\xi_1, \dots, \xi_n, X_{n+1}, \dots, X_m, N_1, \dots, N_n\},$$

where $\{X_{n+1}, \dots, X_m\}$ is an orthogonal basis of $\Gamma(SM|_U)$. Finally, from (30) obtain

$$(32) \quad q = \text{index}(SM)_u + n, \quad \forall u \in M.$$

Hence in this case it follows

P r o p o s i t i o n 2. All screen distributions on a n -lightlike submanifold M of $(\bar{M}, \bar{g})$ have a constant index $q - n$.

C a s e I I I ($1 < r = m < n$). In this case $\text{Rad } TM = TM$ and therefore the tangent bundle of M becomes a vector subbundle of $TM^\perp$. Consider sM as the screen vector bundle of M and obtain

$$(33) \quad TM^\perp = TM \perp sM.$$

Replace Lemma 2 by

L e m m a 4. Let (M, g, sM) be a m -lightlike submanifold of $(\bar{M}, \bar{g})$. Suppose U is a coordinate neighborhood of M and $\{\xi_1, \dots, \xi_m\}$ be a basis of $\Gamma(TM|_U)$. Then there exist smooth sections $\{N_1, \dots, N_m\}$ of $TM|_U$ such that

$$(34) \quad \bar{g}(\bar{N}_i, \xi_j) = \delta_{ij},$$

and

$$(35) \quad \bar{g}(N_i, N_j) = 0, \quad \bar{g}(N_i, W) = 0,$$

for any $i, j \in \{1, \dots, m\}$ and $W \in \Gamma(sM|_U)$.

P r o o f. In this case we set

$$(35') \quad TM|_M = sM \perp sM^\perp,$$

and note that $\text{rank } sM^\perp = 2m$ and TM is a vector subbundle of $sM^\perp$. Consider a complementary vector bundle F of TM in $sM^\perp$ and the proof continues as in Lemma 2.

In this case Theorem 2 is replaced by

T h e o r e m 4. Let (M, g, sM) be a m -lightlike submanifold of $(\bar{M}, \bar{g})$. Then there exists a complementary vector bundle tM of TM in $sM^\perp$ such that $\{N_1, \dots, N_m\}$ from Lemma 4 be a basis of $\Gamma(tM|_U)$.

Therefore we have the decomposition

$$(36) \quad TM|_M = TM \oplus NM = (TM \oplus tM) \perp sM$$

and the local quasi-orthogonal field of frames

$$(37) \quad \{\xi_1, \dots, \xi_m, N_1, \dots, N_m, W_{m+1}, \dots, W_n\},$$

where $\{W_{m+1}, \dots, W_n\}$ is an orthogonal basis of $\Gamma(sM|_U)$. From (36) it follows

$$(38) \quad q = m + \text{index}(sM)_u, \quad \forall u \in M,$$

which enables us to state

P r o p o s i t i o n 3. All screen transversal vector bundles of a m -lightlike submanifold M of $(\bar{M}, \bar{g})$ have a constant index $q - m$.

Case IV ($1 < r = m = n$). In this case $\text{Rad } TM = TM = TM^\perp$ and thus there exist neither a screen distribution nor a screen transversal vector bundle. However, we state

L e m m a 5. Let (M, g) be a m -lightlike submanifold of $(\bar{M}, \bar{g})$. Suppose U is a coordinate neighborhood of M and $\{\xi_1, \dots, \xi_m\}$ be a basis of $\Gamma(TM|_U)$. Then there exist smooth sections $\{N_1, \dots, N_m\}$ of $TM|_U$ such that

$$(39) \quad \bar{g}(N_i, \xi_j) = \delta_{ij},$$

and

$$(40) \quad \bar{g}(N_i, N_j) = 0,$$

for any $i, j \in \{1, \dots, m\}$.

P r o o f. Consider a complementary vector bundle F of TM in $TM|_M$, a basis $\{V_1, \dots, V_m\}$ of $\Gamma(F|_U)$ and the proof follows the same steps as in Lemma 2.

Besides we state

Theorem 5. Let (M, g) be a m -lightlike submanifold of $(\bar{M}, \bar{g})$. Then there exists a complementary vector bundle tM of TM in $TM|_M$ such that $\{N_1, \dots, N_m\}$ from Lemma 5 be a basis of $\Gamma(tM|_M)$.

Hence we have

$$(41) \quad \bar{TM}|_M = TM \oplus tM,$$

and the local quasi-orthogonal field of frames

$$(42) \quad \{\xi_1, \dots, \xi_m, N_1, \dots, N_m\}.$$

As $T_u M \oplus (tM)_u$ is a hyperbolic space for any $u \in M$ we conclude $q = m = n = r$.

Examples. 1. Consider the surface M in $\mathbb{R}_2^4$ given by the equations

$$x^3 = \frac{1}{2}(x^1 + x^2), \quad x^4 = \frac{1}{2} \ln(1 + (x^1 - x^2)^2).$$

Then $TM = \text{Spann}\{U_1, U_2\}$ and $TM^\perp = \text{Spann}\{H_1, H_2\}$ where we set

$$U_1 = \sqrt{2}(1 + (x^1 - x^2)^2) \frac{\partial}{\partial x^1} + (1 + (x^1 - x^2)^2) \frac{\partial}{\partial x^3} + \sqrt{2}(x^1 - x^2) \frac{\partial}{\partial x^4},$$

$$U_2 = \sqrt{2}(1 + (x^1 - x^2)^2) \frac{\partial}{\partial x^2} + (1 + (x^1 - x^2)^2) \frac{\partial}{\partial x^3} + \sqrt{2}(x^2 - x^1) \frac{\partial}{\partial x^4}$$

and

$$H_1 = \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} + \sqrt{2} \frac{\partial}{\partial x^3},$$

$$H_2 = 2(x^2 - x^1) \frac{\partial}{\partial x^2} + \sqrt{2}(x^2 - x^1) \frac{\partial}{\partial x^3} + (1 + (x^1 - x^2)^2) \frac{\partial}{\partial x^4},$$

respectively. Then $\text{Rad } TM = TM \cap TM^\perp$ is a distribution of rank 1 on M spanned by $\xi = H_1$. Hence M is a lightlike submanifold of $\mathbb{R}_2^4$ which falls in Case I. As U_2 and H_2 are timelike and spacelike respectively, we take SM and sM spanned by U_2 and H_2 . Then it follows the lightlike transversal vector bundle tM spanned by

$$N = -\frac{1}{2} \frac{\partial}{\partial x^1} + \frac{1}{2} \frac{\partial}{\partial x^2} + \frac{1}{\sqrt{2}} \frac{\partial}{\partial x^3}.$$

2. Consider in $\mathbb{R}_2^5$ the submanifold M given by the equations

$$x^2 = \{(x^3)^2 + (x^5)^2\}^{1/2}, \quad x^4 = x^1, \quad x^3 > 0, \quad x^5 > 0.$$

Then $TM = \text{Spann}\{U_1, U_2, U_3\}$ and $TM^\perp = \text{Spann}\{\xi_1, \xi_2\}$ where we set

$$\left\{ U_1 = \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^4}, \quad U_2 = x^3 \frac{\partial}{\partial x^2} + x^2 \frac{\partial}{\partial x^3}, \quad U_3 = x^5 \frac{\partial}{\partial x^2} + x^2 \frac{\partial}{\partial x^5} \right\}$$

and

$$\left\{ \xi_1 = \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^4}, \quad \xi_2 = x^2 \frac{\partial}{\partial x^2} + x^3 \frac{\partial}{\partial x^3} + x^5 \frac{\partial}{\partial x^5} \right\},$$

respectively. It follows $\text{Rad } TM = TM^\perp$. Hence M is a 2-lightlike submanifold and falls in Case II. Take SM as being spanned by U_3 and the complementary vector bundle F of $TM^\perp$ in $SM^\perp$ (cf. proof of Lemma 3) as being spanned by $\{V_1 = \partial/\partial x^1, V_2 = \partial/\partial x^3\}$. Then by direct calculations using (11) and (13) obtain

$$B_{11} = -1, \quad B_{12} = B_{21} = 0, \quad B_{22} = \frac{1}{x^3}$$

and

$$A_{11} = \frac{1}{2}, \quad A_{12} = A_{21} = 0, \quad A_{22} = -\frac{1}{2(x^3)^2}.$$

Thus from (10) it follows that the canonical lightlike transversal vector bundle tM is globally spanned by

$$\left\{ N_1 = \frac{1}{2} \left[\frac{\partial}{\partial x^4} - \frac{\partial}{\partial x^1} \right], \quad N_2 = \frac{1}{2(x^3)^2} \left[-x^2 \frac{\partial}{\partial x^2} + x^3 \frac{\partial}{\partial x^3} - x^5 \frac{\partial}{\partial x^5} \right] \right\}.$$

3. Suppose M is a submanifold of $\mathbb{R}_2^5$ given by equations

$$x^3 = \frac{1}{\sqrt{2}}(x^2 + \sin x^1), \quad x^4 = \frac{1}{\sqrt{2}}(x^2 - \sin x^1), \quad x^5 = -\cos x^1.$$

Then we have

$$TM = \text{Spann} \left\{ \begin{aligned} \xi_1 &= \frac{\partial}{\partial x^1} + \frac{\cos x^1}{\sqrt{2}} \frac{\partial}{\partial x^3} - \frac{\cos x^1}{\sqrt{2}} \frac{\partial}{\partial x^4} + \sin x^1 \frac{\partial}{\partial x^5}, \\ \xi_2 &= \frac{\partial}{\partial x^2} + \frac{1}{\sqrt{2}} \frac{\partial}{\partial x^3} + \frac{1}{\sqrt{2}} \frac{\partial}{\partial x^4} \end{aligned} \right\},$$

$$TM^\perp = \text{Spann} \left\{ U_1 = \frac{\cos x^1}{\sqrt{2}} \frac{\partial}{\partial x^1} + \frac{1}{\sqrt{2}} \frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3}, \right. \\ \left. U_2 = -\frac{\cos x^1}{\sqrt{2}} \frac{\partial}{\partial x^1} + \frac{1}{\sqrt{2}} \frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^4}, \quad U_3 = \sin x^1 \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^5} \right\}.$$

Take

$$sM = \text{Spann} \left\{ H_1 = \sin x^1 \frac{\partial}{\partial x^3} - \sin x^1 \frac{\partial}{\partial x^4} - \sqrt{2} \cos x^1 \frac{\partial}{\partial x^5} \right\},$$

and choose the complementary vector bundle F to TM in $sM^\perp$ (cf. Lemma 4) as being

$$F = \text{Spann} \left\{ V_1 = \frac{\partial}{\partial x^1}, \quad V_2 = \frac{\partial}{\partial x^2} \right\}.$$

Then by using (10), (11) and (13) obtain

$$tM = \text{Spann} \left\{ N_1 = \frac{1}{2} \left[-\frac{\partial}{\partial x^1} + \frac{\cos x^1}{\sqrt{2}} \frac{\partial}{\partial x^3} - \frac{\cos x^1}{\sqrt{2}} \frac{\partial}{\partial x^4} + \right. \right. \\ \left. \left. + \sin x^1 \frac{\partial}{\partial x^5} \right], \quad N_2 = \frac{1}{2} \left[-\frac{\partial}{\partial x^2} + \frac{1}{\sqrt{2}} \frac{\partial}{\partial x^3} + \frac{1}{\sqrt{2}} \frac{\partial}{\partial x^4} \right] \right\}.$$

4. Let M be a surface of $\mathbb{R}_2^4$ given by the equations

$$x^3 = F(x^1, x^2), \quad x^4 = G(x^1, x^2),$$

where F and G are smooth on an open and connected domain D of $\mathbb{R}^2$. A straightforward calculation shows that M is 2-lightlike, i.e., M falls in Case IV iff F and G satisfy

$$(F'_{x^1})^2 + (G'_{x^1})^2 = 1, \quad (F'_{x^2})^2 + (G'_{x^2})^2 = 1, \quad F'_{x^1} F'_{x^2} + G'_{x^1} G'_{x^2} = 0.$$

Hence the Jacobian matrix of F and G is orthogonal. Thus there exists a smooth function f such that $F'_{x^1} = G'_{x^2} = \cos f$ and $F'_{x^2} = -G'_{x^1} = \sin f$. Take partial derivat-

ives of these relations and obtain $f = \text{constant}$ on D . Hence the only 2-lightlike surfaces of $\mathbb{R}_2^4$ are portions of lightlike planes.

Received December 20, 1993

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FIBRATUL VECTORIAL TRANSVERSAL AL UNEI SUBVARIETĂȚI LUMINOASE

(Rezumat)

Se construiește un fibrat vectorial transversal pentru orice subvarietate luminoasă a unei varietăți semi-riemanniene. Folosind această construcție se dezvoltă o teorie a subvarietăților luminoase (degenerate) care apare ca o necesitate în teoria relativității.

and of their relation and contact on U. Hence the only significant
 aspect of U. are various of its other parts.

Journal of the American Medical Association
 Chicago, Illinois

Published weekly, 1935

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EDITORIAL: THE AMERICAN MEDICAL ASSOCIATION'S POSITION ON THE ISSUE OF THE DAY

(Continued)

The American Medical Association's position on the issue of the day is a subject of great importance to the medical profession and to the public. It is a subject that has been discussed for many years and is one that will continue to be discussed for many years to come. The American Medical Association's position on this issue is a subject that is of great importance to the medical profession and to the public. It is a subject that has been discussed for many years and is one that will continue to be discussed for many years to come.

FAMILLES DE CONNEXIONS LINÉAIRES ATTACHÉES À UNE CONNEXION SPINORIELLE (I)

PAR

ELENA PUȘCAȘU

1. Introduction. Soit V_4 la variété espace-temps de la relativité générale, $g_{\alpha\beta}$, $(\alpha, \beta, \gamma, \dots = 0, 1, 2, 3)$, le tenseur métrique de la variété et $p_\alpha = (p_{\alpha,b}^a)$, $(a, b, c, \dots = 1, 2, 3)$ un système de matrices Dirac; p_α étant matrice Dirac, nous avons

$$(1) \quad p_\alpha p_\beta + p_\beta p_\alpha = -2g_{\alpha\beta} e,$$

e — la matrice unité d'ordre 4.

Les matrices Dirac p_α définissent un spin-tenseur (spineur de type (1, 1), tenseur de type (0, 1)) nommé le spin-tenseur fondamental de la variété V_4 .

Si ω est 1-forme de la connexion riemannienne sur V_4 et σ est 1-forme de la connexion spinorielle, la liaison entre les deux connexions est donnée par

$$(2) \quad \sigma_{b\rho}^a = -\frac{1}{4} \omega_{\beta\rho}^\alpha p_{\alpha c}^\beta p_b^c, \quad \text{où} \quad p^\beta = g^{\beta\alpha} p_\alpha.$$

Soit $\bar{\nabla}$ le symbole de la dérivée covariante par rapport au couple des connexions (σ, ω) . Nous avons

$$(3) \quad \bar{\nabla}_\alpha p_{\beta b}^a = 0.$$

Pour des détails voir [4].

2. Les matrices Dirac h_α . Ensuite, nous supposons que sur V_4 nous avons deux structures presque complexes déterminées par les tenseurs φ et τ qui

satisfont aux conditions

$$(4) \quad \varphi_{\rho}^{\alpha} \varphi_{\beta}^{\rho} = -\delta_{\beta}^{\alpha}, \quad \tau_{\rho}^{\alpha} \tau_{\beta}^{\rho} = -\delta_{\beta}^{\alpha}, \quad \varphi_{\rho}^{\alpha} \tau_{\beta}^{\rho} + \tau_{\rho}^{\alpha} \varphi_{\beta}^{\rho} = 0.$$

On considère les matrices h_{α} définies par

$$(5) \quad h_{\alpha} = F_{\alpha}^{\beta}(\lambda, \mu) p_{\beta},$$

où $F_{\alpha}^{\beta}(\lambda, \mu)$ est donné par

$$(6) \quad F_{\alpha}^{\beta}(\lambda, \mu) = \lambda \varphi_{\alpha}^{\beta} + \mu \tau_{\alpha}^{\beta}, \quad \lambda, \mu - \text{constantes}.$$

De (1), (4), (5) et (6) on obtient le

T h é o r è m e 1. Une condition nécessaire et suffisante pour que les matrices h_{α} forment un système de matrices Dirac et que nous ayons

$$(7) \quad \lambda^2 + \mu^2 = 1.$$

De (4), (6) et (7) il résulte le

T h é o r è m e 2. Une condition nécessaire et suffisante pour que les matrices h_{β} forment un système de matrices Dirac est que le tenseur $F_{\alpha}^{\beta}(\lambda, \mu)$ satisfasse à la condition

$$(8) \quad F_{\alpha}^{\rho}(\lambda, \mu) F_{\rho}^{\beta}(\lambda, \mu) = -\delta_{\alpha}^{\beta}.$$

Donc, nous avons sur V_4 une autre structure presque complexe définie par le tenseur F .

3. Connexions linéaires attachées à une connexion spinorielle. Étant donnée la connexion spinorielle σ , nous nous proposons ensuite de déterminer la connexion linéaire Γ , de sorte que la dérivée covariante du spin-tenseur $h_{\beta b}^a$ par rapport au couple de connexions (σ, Γ) soit nulle.

Si ∇ est le symbole de la dérivée covariante par rapport au couple (σ, Γ) nous avons

$$(9) \quad \nabla_{\alpha} h_{\beta b}^a = \partial_{\alpha} h_{\beta b}^a + \sigma_{c\alpha}^a h_{\beta b}^c - \sigma_{b\alpha}^c h_{\beta c}^a - \Gamma_{\beta\alpha}^{\nu} h_{\nu b}^a = 0.$$

De (2), (5) et (9) résulte

$$(10) \quad \Gamma_{\beta\alpha}^{\nu}(\lambda, \mu) = \omega_{\beta\alpha}^{\nu} + F_{\beta}^{\rho}(\lambda, \mu) \overset{\circ}{\nabla}_{\alpha} F_{\rho}^{\nu}(\lambda, \mu) \quad \text{avec} \quad \lambda^2 + \mu^2 = 1,$$

où $\overset{\circ}{\nabla}$ est le symbole de la dérivée covariante par rapport à la connexion ω .

Nous obtenons ainsi, une famille de connexions linéaires $G(\lambda, \mu)$ des éléments $\Gamma(\lambda, \mu)$ donnés par (10).

En utilisant le même procédé comme dans [5], on montre que sur $G(\lambda, \mu)$ peut être introduite une opération par rapport à laquelle $G(\lambda, \mu)$ a une

structure de groupe commutatif avec $\Gamma_{\beta\alpha}^{\nu}(0,1)=\omega_{\beta\alpha}^{\nu}+\tau_{\beta}^{\rho}\overset{\circ}{\nabla}_{\alpha}\tau_{\rho}^{\nu}$ — élément neutre et $\Gamma(-\lambda, \mu)$ — élément symétrique de l'élément $\Gamma(\lambda, \mu)$.

4. Certaines propriétés de la connexion $\Gamma(\lambda, \mu)$.

P_1 . Γ est une connexion métrique.

Vraiment, un calcul simple nous donne

$$(11) \quad \nabla_{\alpha}g_{\beta\gamma}=\partial_{\alpha}g_{\beta\gamma}-\Gamma_{\beta\alpha}^{\rho}g_{\rho\gamma}-\Gamma_{\gamma\alpha}^{\rho}g_{\beta\rho}=0.$$

P_2 . Γ est F -connexion si et seulement si ω est F -connexion.

Vraiment, nous avons

$$(12) \quad \nabla_{\alpha}F_{\beta}^{\rho}=-\overset{\circ}{\nabla}_{\alpha}F_{\beta}^{\rho}.$$

Donc

$$\nabla_{\alpha}F_{\beta}^{\rho}=0 \quad \text{si et seulement si} \quad \overset{\circ}{\nabla}_{\alpha}F_{\beta}^{\rho}=0.$$

P_3 . Soit $N_{\beta\alpha}^{\nu}$ le tenseur de torsion de la structure presque complexe F (le tenseur de Nijenhuis) et $T_{\beta\alpha}^{\nu}$ le tenseur de torsion de la connexion Γ . Nous avons

$$(13) \quad N_{\beta\alpha}^{\rho}F_{\rho}^{\nu}F_{\tau}^{\alpha}=\Phi_3T_{\beta\tau}^{\nu},$$

où Φ_3 est l'opérateur Obata défini par

$$(14) \quad \Phi_3T_{\beta\tau}^{\nu}=\frac{1}{2}\left(T_{\beta\tau}^{\nu}-F_{\beta}^{\alpha}F_{\tau}^{\rho}T_{\alpha\rho}^{\nu}\right).$$

Il en résulte le

Théorème 3. Une condition nécessaire et suffisante pour que la structure presque complexe F soit intégrable est que le tenseur de torsion de la connexion Γ vérifie la condition

$$(15) \quad \Phi_3T=0.$$

P_4 . Le couple de connexions (ω, Γ) est compatible avec la structure presque complexe F [2].

Vraiment, si le vecteur v^{α} se transporte par parallélisme sur une courbe arbitraire de V_4 par rapport à la connexion Γ , le vecteur $u^{\alpha}=F_{\beta}^{\alpha}v^{\beta}$ se transporte par parallélisme par rapport à la connexion ω sur la même courbe.

P_5 . Les tenseurs de courbure de deux connexions Γ et ω sont liés par

$$(16) \quad R_{\alpha\beta\rho}^{\varepsilon}=-F_{\alpha}^{\theta}F_{\tau}^{\varepsilon}p_{\theta\beta\rho}^{\tau}.$$

Donc, Γ est sans courbure si et seulement si ω est sans courbure.

5. Les invariants du groupe $G(\lambda, \mu)$. Les tenseurs de torsion et de courbure de la connexion Γ sont donnés par

$$(17) \quad T_{\alpha\beta}^{\nu} = F_{\rho}^{\nu} \left(\overset{\circ}{\nabla}_{\alpha} F_{\beta}^{\rho} - \overset{\circ}{\nabla}_{\beta} F_{\alpha}^{\rho} \right),$$

$$(18) \quad R_{\alpha\beta\rho}^{\varepsilon} = r_{\alpha\beta\rho}^{\varepsilon} + F_{\alpha}^{\nu} \left(\overset{\circ}{\nabla}_{\rho} \overset{\circ}{\nabla}_{\beta} F_{\nu}^{\varepsilon} - \overset{\circ}{\nabla}_{\beta} \overset{\circ}{\nabla}_{\rho} F_{\nu}^{\varepsilon} \right),$$

où $r_{\alpha\beta\rho}^{\varepsilon}$ est le tenseur de courbure de la connexion ω .

À l'aide de ces tenseurs on peut obtenir les invariants suivants du groupe $G(\lambda, \mu)$:

$$(19) \quad I_{1\alpha\beta}^{\nu} = \frac{1}{\nu} \left[T_{\alpha\beta}^{\theta} F_{\theta}^{\rho} \varphi_{\rho}^{\nu} + \mu \varphi_{\eta}^{\nu} \left(\overset{\circ}{\nabla}_{\alpha} \tau_{\beta}^{\eta} - \overset{\circ}{\nabla}_{\beta} \tau_{\alpha}^{\eta} \right) \right] = \varphi_{\eta}^{\nu} \left(\overset{\circ}{\nabla}_{\beta} \varphi_{\alpha}^{\eta} - \overset{\circ}{\nabla}_{\alpha} \varphi_{\beta}^{\eta} \right),$$

$$(20) \quad I_{2\alpha\beta}^{\nu} = \frac{1}{\mu} \left[T_{\alpha\beta}^{\theta} F_{\theta}^{\rho} \tau_{\rho}^{\nu} + \lambda \tau_{\eta}^{\nu} \left(\overset{\circ}{\nabla}_{\alpha} \varphi_{\beta}^{\eta} - \overset{\circ}{\nabla}_{\beta} \varphi_{\alpha}^{\eta} \right) \right] = \tau_{\eta}^{\nu} \left(\overset{\circ}{\nabla}_{\beta} \tau_{\alpha}^{\eta} - \overset{\circ}{\nabla}_{\alpha} \tau_{\beta}^{\eta} \right),$$

$$(21) \quad I_{3\eta\beta\rho}^{\varepsilon} = \frac{1}{\lambda} \left[R_{\alpha\beta\rho}^{\varepsilon} F_{\tau}^{\alpha} \varphi_{\eta}^{\tau} - \mu \varphi_{\eta}^{\tau} \tau_{\nu}^{\varepsilon} r_{\tau\beta\rho}^{\nu} \right] = \varphi_{\eta}^{\tau} \varphi_{\alpha}^{\varepsilon} r_{\tau\beta\rho}^{\alpha},$$

$$(22) \quad I_{4\eta\beta\rho}^{\varepsilon} = \frac{1}{\mu} \left[R_{\alpha\beta\rho}^{\varepsilon} F_{\tau}^{\alpha} \tau_{\eta}^{\tau} - \lambda \tau_{\eta}^{\tau} \varphi_{\nu}^{\varepsilon} r_{\tau\beta\rho}^{\nu} \right] = \tau_{\eta}^{\tau} \tau_{\alpha}^{\varepsilon} r_{\tau\beta\rho}^{\alpha}.$$

Une interprétation géométrique des invariants I_1, I_2, I_3, I_4 est donnée par les théorèmes suivants:

Théorème 4. Il y a une seule connexion du groupe $G(\lambda, \mu)$, $\Gamma(1, 0)$ qui ait comme tenseur de torsion l'invariant I_1 et comme tenseur de courbure l'invariant I_3 .

Théorème 5. Il y a une seule connexion du groupe $G(\lambda, \mu)$, $\Gamma(0, 1)$ qui ait comme tenseur de torsion l'invariant I_2 et comme tenseur de courbure l'invariant I_4 .

Reçu le 29 Avril, 1993

Université Technique "Gh. Asachi", Jassy,
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FAMILII DE CONEXIUNI LINIARE ATAȘATE UNEI CONEXIUNI SPINORIALE (I)

(Rezumat)

Pe varietatea spațiu-timp V_4 din relativitatea generală se construiește, cu ajutorul a două structuri aproape complexe, un sistem de matrice Dirac h_α . Considerând conexiunea spinorială σ dată pe V_4 , se determină toate conexiunile liniare Γ astfel ca derivata covariantă a spin-tensorului $h_{\alpha b}^a$ în raport cu perechea (σ, Γ) să fie nulă. Mulțimea acestor conexiuni are structură de grup. Se determină apoi o serie de invarianți ai acestui grup și se dau unele interpretări geometrice ale lor.

1. The purpose of this study is to determine the effect of the intervention on the outcome.
 2. The study was approved by the Institutional Review Board on 11/10/2014.
 3. The study was approved by the Institutional Review Board on 11/10/2014.
 4. The study was approved by the Institutional Review Board on 11/10/2014.

INSTITUTIONAL REVIEW BOARD

(Continued)

5. The study was approved by the Institutional Review Board on 11/10/2014.
 6. The study was approved by the Institutional Review Board on 11/10/2014.
 7. The study was approved by the Institutional Review Board on 11/10/2014.
 8. The study was approved by the Institutional Review Board on 11/10/2014.

REGULAR MAPPINGS IN FINSLER SPACES

II. CHARACTERIZATIONS OF QUASICONFORMALITY IN VECTORIAL NORMED SPACES

BY

VERONICA T. BORCEA and AL. NEAGU

In this Note we give theorems of characterization for quasiconformality in vectorial normed spaces. These results will be applied to the study of the regular mappings between Finsler spaces.

If D is a domain in the vectorial normed space V , then the inclusion map, $i: D \rightarrow V$, defines a global chart on D . The tangent space at $x \in D$ is $T_x D = \{x\} \times V$ and the tangent bundle is $TD = D \times V$. For every $x \in D$, there exists a ball $B(O_x, r) \subset T_x D$, $O_x = (x, 0) \in T_x D$ such that the map $t_x: B(O_x, r) \rightarrow D$, $t_x(X_x) = x + X$, where $X_x = (x, X)$ defines a chart $t_x^{-1}: B(x, r) \subset D \rightarrow B(O_x, r) \subset T_x D$. The tangent space $T_x D$ has a vectorial normed space structure induced by V and thus t_x is an isometry.

In the following we consider $f: D \rightarrow \tilde{D}$ ($D, \tilde{D}$ domains in vectorial normed spaces) a differentiable Fréchet mapping at $x \in D$. We define $T_x f = DF_x(O_x)$, where $F_x = t_x^{-1} \circ f \circ t_x$, $\tilde{x} = f(x)$ and $DF_x(O_x)$ denotes the Fréchet derivative of F_x at O_x . We obtain

$$\begin{aligned} (T_x f)X_x &= DF_x(O_x)X_x = \lim_{\lambda \rightarrow 0} \frac{F_x(O_x + \lambda X_x) - F_x(O_x)}{\lambda} = \\ &= \lim_{\lambda \rightarrow 0} \frac{(t_x^{-1} \circ f)(x + \lambda X) - (t_x^{-1} \circ f)(x)}{\lambda} = \lim_{\lambda \rightarrow 0} \frac{(\tilde{x}, f(x + \lambda X) - \tilde{x}) - (\tilde{x}, f(x) - \tilde{x})}{\lambda} = \\ &= \lim_{\lambda \rightarrow 0} \frac{(\tilde{x}, f(x + \lambda X) - \tilde{x} - f(x) + \tilde{x})}{\lambda} = \left[\tilde{x}, \lim_{\lambda \rightarrow 0} \frac{f(x + \lambda X) - f(x)}{\lambda} \right] = (\tilde{x}, Df(x)X) = \\ &= (Df(x)X)_x. \end{aligned}$$

The map $\delta_f: D \rightarrow \mathbb{R}$ defined by

$$(1) \quad \delta_f(x) = \limsup_{r \rightarrow 0} \frac{L(x, r)}{l(x, r)},$$

where

$$(2) \quad \begin{aligned} L(x, r) &= \sup \{ \|f(y) - f(x)\| / \|y - x\| = r \}, \\ l(x, r) &= \inf \{ \|f(y) - f(x)\| / \|y - x\| = r \}, \end{aligned}$$

is called the *linear dilatation* of f .

A homeomorphism $f: D \rightarrow \tilde{D}$ is called *K-quasiconformal* (K-qc), $1 \leq K < \infty$, if

$$(3) \quad \delta_f(x) \leq K, \quad \text{for every } x \in D.$$

If f is a C -isometry, then $\delta_f(x) \leq C^2$, for every $x \in D$, hence f is C^2 -qc.

The maps $D^+f, D^-f: D \rightarrow \mathbb{R}$ defined by

$$(4) \quad \begin{aligned} D^+f(x) &= \limsup_{y \rightarrow x} \frac{\|f(y) - f(x)\|}{\|y - x\|}, \\ D^-f(x) &= \liminf_{y \rightarrow x} \frac{\|f(y) - f(x)\|}{\|y - x\|}, \end{aligned}$$

are called the *upper scalar derivative* and the *lower scalar derivative*, respectively, of f . We have generally

$$0 \leq D^-f(x) \leq D^+f(x) \leq \infty.$$

If $D^-f(x) = D^+f(x) = 0$ or ∞ , we say that f is *degenerate* at x .

The *modulus of dilatation* of the map f in a direction S is

$$\left| \frac{\partial f(x)}{\partial S} \right| = \limsup_{\lambda \rightarrow 0} \left\| \frac{f(x + \lambda S) - f(x)}{\lambda} \right\|.$$

We have

$$(5) \quad \begin{aligned} \left| \frac{\partial F_x(O_x)}{\partial S} \right| &= \limsup_{\lambda \rightarrow 0} \left\| \frac{F_x(O_x + \lambda S) - F_x(O_x)}{\lambda} \right\| = \\ &= \limsup_{\lambda \rightarrow 0} \left\| \frac{(t_x^{-1} \circ f)(x + \lambda S) - (t_x^{-1} \circ f)(x)}{\lambda} \right\| = \limsup_{\lambda \rightarrow 0} \left\| \frac{t_x^{-1}(f(x + \lambda S)) - t_x^{-1}(f(x))}{\lambda} \right\| = \\ &= \limsup_{\lambda \rightarrow 0} \left\| \frac{(\tilde{x}, f(x + \lambda S) - f(x)) - (\tilde{x}, 0)}{\lambda} \right\| = \limsup_{\lambda \rightarrow 0} \left\| \frac{(\tilde{x}, f(x + \lambda S) - f(x))}{\lambda} \right\| = \\ &= \limsup_{\lambda \rightarrow 0} \left\| \frac{f(x + \lambda S) - f(x)}{\lambda} \right\| = \left| \frac{\partial f(x)}{\partial S} \right|. \end{aligned}$$

L e m m a 1. If $f: D \rightarrow \tilde{D}$ is differentiable at $x \in D$, then

$$(6) \quad D^+f(x) = \sup_S \left| \frac{\partial f(x)}{\partial S} \right|, \quad D^-f(x) = \inf_S \left| \frac{\partial f(x)}{\partial S} \right|.$$

P r o o f. By absurd, we suppose that $k = \sup_S \left| \frac{\partial f(x)}{\partial S} \right| < D^+f(x) = k + \alpha, \alpha > 0$.

Then there exists the sequence $(x_n), x_n \rightarrow x$, such that

$$\lim_{n \rightarrow \infty} \frac{\|f(x_n) - f(x)\|}{\|x_n - x\|} = k + \alpha,$$

$$\begin{aligned} \frac{\|f(x_n) - f(x)\|}{\|x_n - x\|} &= \frac{\|Df(x)(x_n - x) + \varepsilon(x)(x_n - x)\|}{\|x_n - x\|} \leq \\ &\leq \frac{\|Df(x)(x_n - x)\|}{\|x_n - x\|} + \|\varepsilon(x)(x_n - x)\|. \end{aligned}$$

We choose n sufficiently large, such that

$$(7) \quad \frac{\|f(x_n) - f(x)\|}{\|x_n - x\|} > k + \frac{\alpha}{2} \quad \text{and} \quad \|\varepsilon(x)(x_n - x)\| < \frac{\alpha}{2}$$

and let $S = (x_n - x) / \|x_n - x\|$, hence $x_n - x = tS, \|S\| = 1$. Then

$$\frac{\|f(x_n) - f(x)\|}{\|x_n - x\|} \leq \|Df(x)S\| + \|\varepsilon(x)(tS)\|.$$

But

$$\begin{aligned} \|Df(x)S\| &= \left\| \lim_{t \rightarrow 0} \frac{f(x + tS) - f(x)}{t} \right\| \leq \lim_{t \rightarrow 0} \sup \frac{\|f(x + tS) - f(x)\|}{|t|} = \\ &= \left| \frac{\partial f(x)}{\partial S} \right| \leq \sup_S \left| \frac{\partial f(x)}{\partial S} \right| = k. \end{aligned}$$

We have

$$\frac{\|f(x_n) - f(x)\|}{\|x_n - x\|} \leq k + \frac{\alpha}{2}$$

contradicting (7). It follows that $\sup_S \left| \frac{\partial f(x)}{\partial S} \right| \geq D^+f(x)$ and because the inverse inequality is evident, we have the first relation in (6).

Analogously we obtain the second equality in (6).

We have

$$D^+f(x) = \sup_S \left| \frac{\partial f(x)}{\partial S} \right| = \sup_S \left| \frac{\partial F_x(O_x)}{\partial S} \right| = D^+F_x(O_x).$$

Analogously we obtain $D^-f(x) = D^-F_x(O_x)$.

If f is differentiable at x , then

$$D^+F_x(O_x) = \sup_{\|X_x\|=1} \|DF_x(O_x)X_x\| = \|DF_x(O_x)\|,$$

$$D^-F_x(O_x) = \inf_{\|X_x\|=1} \|DF_x(O_x)X_x\|.$$

We obtain successively

$$(8) \quad T_x f = \sup_{\|X_x\|=1} \|(T_x f)X_x\| = \sup_{\|X_x\|=1} \|DF_x(O_x)X_x\| = D^+F_x(O_x) = D^+f(x).$$

Analogously we obtain

$$(9) \quad \inf_{\|X_x\|=1} \|(T_x f)X_x\| = D^-f(x).$$

If $T_x f$ is bijective, then

$$(10) \quad D^-f(x) = \|(T_x f)^{-1}\|^{-1}.$$

In this case we have (see [2])

$$(11) \quad p_1(O_x) = \|T_x f\| \|(T_x f)^{-1}\| = \frac{D^+f(x)}{D^-f(x)}.$$

Let $\sigma = \{\sigma_\alpha\}$, $\alpha \in (0, 1)$, be a regular hypersurface family of parameter $k_\alpha(x)$ relative to x (see [2]). Then the family of hypersurfaces $\tau = \{\tau_\alpha / \tau_\alpha = t_x^{-1}(\sigma_\alpha)\} = \{(x, y-x) / y \in \sigma_\alpha\}$ is regular of parameter $k_\sigma(O_x) = k_\sigma(x)$. Indeed

$$\begin{aligned} k_\tau(O_x) &= \lim_{\alpha \rightarrow 0} \sup \frac{\sup\{\|(x, y-x) - (x, 0)\| / (x, y-x) \in \tau_\alpha\}}{\inf\{\|(x, y-x) - (x, 0)\| / (x, y-x) \in \tau_\alpha\}} = \\ &= \lim_{\alpha \rightarrow 0} \sup \frac{\sup\{\|y-x\| / y \in \sigma_\alpha\}}{\inf\{\|y-x\| / y \in \sigma_\alpha\}} = k_\sigma(x). \end{aligned}$$

We denote $\bar{\sigma} = \{\bar{\sigma}_\alpha / \bar{\sigma}_\alpha = f(\sigma_\alpha)\}$, $\tau = \{\tau_\alpha / \tau_\alpha = (T_x f)(\sigma_\alpha)\}$, $\bar{\tau} = \{\bar{\tau}_\alpha / \bar{\tau}_\alpha = F_x(\sigma_\alpha)\}$. It follows that $\bar{\tau} = t_x^{-1}(\bar{\sigma})$, hence, if $\bar{\tau}'$ and $t_x^{-1}(\bar{\sigma})$ are regular, then $\bar{k}_{\bar{\tau}}(O_{\bar{x}}) = \bar{k}_{\bar{\sigma}}(\bar{x})$.

L e m m a 2. If $f: D \rightarrow \bar{D}$ is a differentiable homeomorphism with $T_x f$

bijjective then f maps regular hypersurface family σ of parameter $k_\sigma(x)$ into a regular family $\tilde{\sigma}$ of parameter $\tilde{k}_\sigma(\tilde{x})$ iff $T_x f$ maps the regular hypersurface family τ into a regular family $\tilde{\tau}$ of parameter $\tilde{k}_\tau(O_x) = \tilde{k}_\tau(\tilde{x})$.

P r o o f. The following three assertions are equivalent:

- a) f maps σ of parameter $k_\sigma(x)$ into $\tilde{\sigma}$ of parameter $\tilde{k}_\sigma(\tilde{x})$;
- b) F_x maps τ of parameter $k_\tau(O_x) = k_\sigma(x)$ into $\tilde{\tau}$ of parameter $\tilde{k}_\tau(O_x) = \tilde{k}_\tau(\tilde{x})$;
- c) $T_x f = DF_x(O_x)$ maps τ of parameter $k_\tau(O_x)$ into $\tilde{\tau}$ of parameter $\tilde{k}_\tau(O_x) = \tilde{k}_\tau(\tilde{x})$.

The equivalence between b) and c) it follows from Theorem 2 of [2].

Remark 1. By the hypotheses of Lemma 2 we have

$$(12) \quad q_f(x) = q_{T_x f}(O_x) = p_1(O_x),$$

where $q_f(x) = \inf_{\sigma} k_\sigma(x) \tilde{k}_\sigma(\tilde{x})$ (see Theorem 1 from [2]).

Theorem 1. A diffeomorphism $f: D \rightarrow \tilde{D}$ is K -qc iff $q_f \leq K$, for every $x \in D$.

P r o o f. Let f be a diffeomorphism K -qc in D and $\sigma = \{\sigma_\alpha / \sigma_\alpha = S(x, \alpha) \subset D, \alpha \in (0, 1)\}$; evidently $k_\sigma(x) = 1$. We have $\tilde{\sigma} = \{\tilde{\sigma}_\alpha / \tilde{\sigma}_\alpha = f(\sigma_\alpha)\}$ and $\tilde{k}_\sigma(\tilde{x}) = \delta_f(x) \leq K$, for every $x \in D$. It follows that $q_f(x) = \inf_{\sigma} k_\sigma(x) \tilde{k}_\sigma(\tilde{x}) \leq \delta_f(x) \leq K$, for every $x \in D$.

Conversely, we suppose $q_f(x) \leq K$, for every $x \in D$, then, by (12), $p_1(O_x) \leq K$, for every $x \in D$. Let $S(x, r) \subset D$ be and $x', x'' \in S(x, r)$ such that

$$\|f(x') - f(x)\| > \sup\{\|f(y) - f(x)\| - \varepsilon r, y \in S(x, r)\},$$

$$\|f(x'') - f(x)\| < \inf\{\|f(y) - f(x)\| + \varepsilon r, y \in S(x, r)\}.$$

We obtain

$$\begin{aligned} \delta_f(x) &= \lim_{r \rightarrow 0} \sup \frac{\sup\{\|f(y) - f(x)\| / y \in S(x, r)\}}{\inf\{\|f(y) - f(x)\| / y \in S(x, r)\}} < \\ &< \lim_{r \rightarrow 0} \sup \frac{\|f(x') - f(x)\| + \varepsilon r}{\|f(x'') - f(x)\| - \varepsilon r} \leq \frac{\limsup_{\substack{y \rightarrow x \\ y \in S(x, r)}} \frac{\|f(x) - f(y)\|}{\|x - y\|} + \varepsilon}{\liminf_{\substack{y \rightarrow x \\ y \in S(x, r)}} \frac{\|f(x) - f(y)\|}{\|x - y\|} - \varepsilon} = \frac{D^+ f(x) + \varepsilon}{D^- f(x) - \varepsilon}. \end{aligned}$$

For $\varepsilon \rightarrow 0$ we have $\delta_f(x) \leq D^+ f(x) / D^- f(x) = p_1(O_x) \leq K$, for every $x \in D$, hence f is K -qc in D .

Remark 2. For the necessity it is sufficient that f to be homeomorphism.

Remark 3. If f is a K -qc diffeomorphism, then f^{-1} is K -qc (see [2]).

Remark 4. If $f: D \rightarrow \tilde{D}$ and $g: \tilde{D} \rightarrow \tilde{\tilde{D}}$ are diffeomorphisms K , respectively $\tilde{K}$ -qc, then $g \circ f$ is $K\tilde{K}$ -qc (see [2]).

Let $U: D \times V \rightarrow D \times V$ be an automorphism of a vector bundle (named field of characteristics), hence its restriction U_x to $T_x D$ is an automorphism (of topological vector space) of $T_x D$. The image of the sphere $S(O_x, t)$ by U_x is an ellipsoid $E_t(U_x) = \{Y_x / \|U_x^{-1}(Y_x)\| = t\}$. The family $E_t(U) = \{E_t(U_x) / x \in D\}$ is called field of ellipsoids. For t sufficiently small, $\mathcal{E}_t(U_x) = t_x(E_t(U_x)) \subset D$ and we can consider the family of ellipsoids over D , $\mathcal{E}_t(U) = \{\mathcal{E}_t(U_x) / x \in D\}$. Because t_x is an isometry $\mathcal{E}_t(U_x)$ and $E_t(U_x)$ have the same extreme semiaxes.

A homeomorphism $f: D \rightarrow \tilde{D}$ is called $C\tilde{C}$ -homeomorphism if there exist the fields of characteristics U and $\tilde{U}$ over D , respectively $\tilde{D}$ and a map $\varepsilon: D \rightarrow (0, \infty)$ such that, for every $x \in D$ and for every $t \in (0, \varepsilon(x))$, the image of the ellipsoid $\mathcal{E}_t(U_x)$ by f is a surface Σ_t contained between the ellipsoids $\tilde{\mathcal{E}}_{\tilde{t}_1}(\tilde{U}_x)$, ($i=1, 2$), with $\lim_{t \rightarrow 0} \tilde{t}_1 \tilde{t}_2^{-1} = 1$. If for every $x \in D$, $\tilde{U}_x$ is the identity of $T_x \tilde{D}$, then f is called C -homeomorphism and if for every $x \in D$, U_x is the identity of $T_x D$, then f is called $\tilde{C}$ -homeomorphism.

We observe that, if f is C -homeomorphism and $\tilde{C}$ -homeomorphism, then f is conformal. If f is a linear homeomorphism, then it is $C\tilde{C}$ -homeomorphism, C -homeomorphism and $\tilde{C}$ -homeomorphism.

By Lemma 2 it follows that the diffeomorphism $f: D \rightarrow \tilde{D}$ is $C\tilde{C}$ -homeomorphism with the characteristics U and $\tilde{U}$ iff $T_x f$ maps the field of ellipsoids of characteristic U into the field of ellipsoids of characteristic $\tilde{U}$.

Theorem 2. A diffeomorphism $f: D \rightarrow \tilde{D}$ is K -qc iff f is C -homeomorphism and $p_1(x) \leq K$, for every $x \in D$.

Proof. If f is K -qc, then by Theorem 1 we have that $q_f(x) \leq K$, for every $x \in D$. By Lemma 2 from [2] it follows that f and $T_x f$ have the same characteristic $q(x)$. Because $T_x f$ is linear, it is C -homeomorphism and $q_{T_x f}(x) = p_1(O_x)$ (see Theorem 1 from [2]). It follows that $T_x f$ satisfies the conditions of the theorem, hence f satisfies the conditions of the theorem.

The sufficiency is evident.

Theorem 3. A diffeomorphism $f: D \rightarrow \tilde{D}$ is K -qc iff f is $C\tilde{C}$ -homeomorphism and $p_1(x) \tilde{p}_1(\tilde{x}) \leq K$, for every $x \in D$.

Proof. The necessity is evident. The image of the ellipsoid $\mathcal{E}_t(U_x)$ by f is a surface Σ_t contained between the ellipsoids $\tilde{\mathcal{E}}_{\tilde{t}_1}(\tilde{U}_x)$ and $\tilde{\mathcal{E}}_{\tilde{t}_2}(\tilde{U}_x)$ with $\lim_{t \rightarrow 0} \tilde{t}_1 \tilde{t}_2^{-1} = 1$. The parameter $k(x)$ of the family $\{\mathcal{E}_t(U_x) / t \in (0, \varepsilon(x))\}$ is

$$k(x) = \lim_{t \rightarrow 0} \sup \frac{a_1(x, t)}{a_0(x, t)} = p_1(x).$$

The parameter $\tilde{k}(\bar{x})$ of the family $\{\Sigma_t/t \in (0, \varepsilon(x))\}$ is

$$\tilde{k}(\bar{x}) = \lim_{t \rightarrow 0} \sup \frac{\sup \{\|f(y) - f(x)\| / f(y) \in \Sigma_t\}}{\inf \{\|f(y) - f(x)\| / f(y) \in \Sigma_t\}}.$$

We obtain

$$\begin{aligned} \lim_{t \rightarrow 0} \sup \frac{\bar{a}_1(\bar{x}, \bar{t}_2)}{\bar{a}_0(\bar{x}, \bar{t}_2)} \frac{\bar{a}_1(\bar{x}, \bar{t}_1)}{\bar{a}_1(\bar{x}, \bar{t}_2)} &= \lim_{t \rightarrow 0} \sup \frac{\bar{a}_1(\bar{x}, \bar{t}_1)}{\bar{a}_0(\bar{x}, \bar{t}_2)} \leq \tilde{k}(\bar{x}) \leq \\ &\leq \lim_{t \rightarrow 0} \sup \frac{\bar{a}_1(\bar{x}, \bar{t}_2)}{\bar{a}_0(\bar{x}, \bar{t}_1)} = \lim_{t \rightarrow 0} \sup \frac{\bar{a}_1(\bar{x}, \bar{t}_2)}{\bar{a}_0(\bar{x}, \bar{t}_2)} \frac{\bar{a}_0(\bar{x}, \bar{t}_2)}{\bar{a}_0(\bar{x}, \bar{t}_1)}. \end{aligned}$$

Because $\bar{a}_1(\bar{x}, \bar{t}) = \bar{t} \sqrt{\|U_x^{-1}\|}$ and $\bar{a}_0(\bar{x}, \bar{t}) = \bar{t} / \sqrt{\|U_x\|}$ it follows that $\bar{p}_1(\bar{x}) = \tilde{k}(\bar{x})$ and $q(x) \leq k(x) \tilde{k}(\bar{x}) = p_1(x) \bar{p}_1(\bar{x}) \leq K$, for every $x \in D$. By Theorem 1 we have that f is K -qc in D .

Theorem 4. A diffeomorphism $f: D \rightarrow \bar{D}$ is K -qc iff f is $\bar{C}$ -homeomorphism and $\bar{p}_1(\bar{x}) \leq K$, for every $x \in D$.

Proof. The sufficiency is evident.

Conversely, if f is K -qc, then by Theorem 1 we have that $q_f(x) \leq K$, for every $x \in D$. By the other way $p_1(x) = p_1(O_x) = q_f(x)$ where $p_1(O_x)$ is the characteristic parameter of $T_x f$. the linear map $T_x f$ maps the ellipsoids of parameter $p_1(O_x)$ into the spheres, hence $T_x f$ maps the spheres $S(O_x, t)$ into the ellipsoids $\bar{E}_{\bar{t}}(\bar{U}_{\bar{x}})$ of characteristic parameter $\bar{p}_1(O_{\bar{x}}) = p_1(O_x)$, hence $\bar{p}_1(\bar{x}) = p_1(x)$. It follows that f is $\bar{C}$ -homeomorphism and $\bar{p}_1(\bar{x}) \leq K$, for every $x \in D$.

Theorem 5. A diffeomorphism $f: D \rightarrow \bar{D}$ is K -qc iff for every $S, S' \in T_x D$ we have

$$\left| \frac{\partial f(x)}{\partial S} \right| \leq K \left| \frac{\partial f(x)}{\partial S'} \right| \quad \text{for every } x \in D.$$

Proof. The argument is the same as in [3] with respect to the case of separable Hilbert spaces.

Received January 15, 1993

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APLICAȚII REGULATE ÎN SPAȚII FINSLER

II. Caracterizări ale cvasiconformității în spații vectoriale normate

(Rezumat)

Se dau teoreme de caracterizare pentru cvasiconformitate în spații vectoriale normate. Aceste rezultate vor fi aplicate la studiul aplicațiilor regulate în spații Finsler.

CR-PRODUCT OF QUASI-SASAKIAN MANIFOLD

BY

C. CĂLIN

0. Introduction. The notion of CR-submanifold of a Kähler manifold has been introduced by A. Bejancu and then studied by many people (see references of monographs by K. Yano and M. Kon [10] and A. Bejancu [2]). Later on, the concept has been considered by A. Bejancu and N. Papaghiuc [3], K. Yano and M. Kon [10] and M. Kobayashi [9] for almost contact manifolds. In [6] we started the study of CR-submanifolds of quasi-Sasakian manifolds and in [7] we obtain necessary and sufficient conditions for the leaves of distributions on contact CR-submanifolds of a quasi-Sasakian manifold, be totally geodesic immersed in CR-submanifold or the ambient space.

The purpose of this paper is to study the existence of CR-products on quasi-Sasakian manifolds. Because there exist four distributions of a contact CR-submanifold M of a quasi-Sasakian manifold, we define the notion of $(D, D^\perp \oplus \{\xi\})$ -contact CR-product and the notion $(D \oplus \{\xi\}, D^\perp)$ -contact CR-product, if the distribution $D, D^\perp \oplus \{\xi\}$ and, respectively, $D \oplus \{\xi\}, D^\perp$ are integrable and their leaves are totally geodesic immersed in M . This is different from the Sasakian manifold because in this case the distribution D is never integrable (cf. A. Bejancu [2], p. 104).

1. Preliminaries. Let $\tilde{M}$ be a real $2n+1$ -dimensional differentiable manifold and f, ξ and η be a tensor of the type (1.1), a vector field, and a 1-form, respectively, on $\tilde{M}$ satisfying,

$$f^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad f(\xi) = 0, \quad \eta \odot f = 0,$$

where I is the identity on the tangent bundle $T\tilde{M}$ of $\tilde{M}$. Then, on following Blair [5] we say that $\tilde{M}$ is an almost contact manifold and (f, ξ, η) is the

almost contact structure on $\tilde{M}$.

Throughout the paper, all manifolds and maps are differentiable of class C^∞ . We denote by $\mathcal{F}(\tilde{M})$ the ring of the differentiable function on $\tilde{M}$ and by $\Gamma(E)$ the $\mathcal{F}(\tilde{M})$ -module of the sections of a vector bundle E over $\tilde{M}$. We use the same notations for any other manifold involved in the study.

Now suppose there exists a Riemannian metric g which satisfies

$$g(fX, fY) = g(X, Y) - \eta(X)\eta(Y), \quad \forall X, Y \in \Gamma(T\tilde{M}).$$

Then we say that (f, ξ, η, g) is an *almost contact metric structure* and $\tilde{M}$ is an *almost contact metric manifold*. Also, we recall that the Nijenhuis tensor of f is a tensor field N_f of type (1,1) given by

$$N_f(X, Y) = [fX, fY] + f^2[X, Y] - f[fX, Y] - f[X, fY], \quad \forall X, Y \in \Gamma(T\tilde{M}).$$

We say that the almost contact structure (f, ξ, η) is *normal* if the following condition is satisfied

$$N_f + 2d\eta \otimes \xi = 0.$$

The fundamental 2-form ϕ of $\tilde{M}$ is defined by

$$\phi(X, Y) = g(X, fY), \quad \forall X, Y \in \Gamma(T\tilde{M}).$$

Finally, we say that $\tilde{M}$ is a *quasi-Sasakian manifold* (cf. D. E. Blair [4]) if it is endowed with a normal almost contact metric structure (f, ξ, η, g) and the fundamental 2-form ϕ is closed.

A characterisation of quasi-Sasakian manifold by means of the covariant derivative of f has been given by S. K a n e m a k i [8] as follows: $\tilde{M}$ is a quasi-Sasakian manifold if and only if it is endowed with an almost contact metric structure (f, ξ, η, g) and a tensor field F of the type (1,1) so that

$$(1.1) \quad (\tilde{\nabla}_X f)Y = \eta(Y)FX - g(FX, Y)\xi, \quad fFX = FfX,$$

$$g(FX, Y) = g(X, FY),$$

for any $X, Y \in \Gamma(T\tilde{M})$. By using (1.1) we infer

$$(1.2) \quad \nabla_X \xi = fFX, \quad \forall X \in \Gamma(T\tilde{M}).$$

Let $\tilde{M}$ be a quasi-Sasakian manifold and M a m -dimensional submanifold so that ξ is tangent to M . We denote by $\{\xi\}$ the 1-dimensional distribution spanned by ξ on M . We use the same symbol g for the induced Riemannian metric on M . Then we say that M is a *contact CR-submanifold* of $\tilde{M}$ if there exist two distributions D and $D^\perp$ so that (cf. [3])

$$TM = D \oplus D^\perp \oplus \{\xi\},$$

and

$$fX \in \Gamma(D), \quad fY \in \Gamma(TM^\perp), \quad \forall X \in \Gamma(D), \quad Y \in \Gamma(D^\perp),$$

where $TM^\perp$ is the normal bundle of M . We denote by $\Gamma(\mu)$ the orthogonal

complement of $fD^\perp$ in $\Gamma(TM^\perp)$.

Let M be a contact CR-submanifold of a quasi-Sasakian manifold $\tilde{M}$. Denote by P and Q the projection morphisms of TM on D and $D^\perp$, respectively. Hence we have

$$X = PX + QX + \eta(X)\xi, \quad \forall X \in \Gamma(TM),$$

from which we deduce

$$fX = tX + \omega X, \quad \forall X \in \Gamma(TM),$$

where $tX = fPX$ and $\omega X = fQX$. Now for any $N \in \Gamma(TM^\perp)$ we put

$$(1.3) \quad fN = BN + CN,$$

where BN and CN are the tangent part and the normal part of fN , respectively.

Finally, we recall the Gauss and Weingarten formulae

$$(1.4) \quad \nabla_X Y = \nabla_X Y + h(X, Y), \quad \forall X, Y \in \Gamma(TM),$$

and

$$(1.5) \quad \tilde{\nabla}_X Y = -A_N X + \nabla_X^\perp N, \quad \forall X \in \Gamma(TM), \quad N \in \Gamma(TM^\perp),$$

where $\tilde{\nabla}, \nabla$ and $\nabla^\perp$ are connections on $\tilde{M}, TM$ and $TM^\perp$ respectively, and A_N is the shape operator with respect to N , h is the second fundamental form of immersion $M \hookrightarrow \tilde{M}$.

Next we recall some results from [7] for later use.

Theorem 1.1. *Let M be a contact CR-submanifold of a quasi-Sasakian manifold $\tilde{M}$. Then we have:*

i) *The distribution D is integrable and its leaves are totally geodesic immersed in M if and only if*

$$(1.6) \quad h(X, Y) \in \Gamma(\mu), \quad \forall X, Y \in \Gamma(D),$$

and

$$(1.7) \quad FD \perp D.$$

ii) *The distribution D is integrable and its leaves are totally geodesic immersed in $\tilde{M}$ if and only if (2.2) is satisfied and*

$$(1.8) \quad h(XY) = 0, \quad \forall X, Y \in \Gamma(D).$$

Theorem 1.2. *Let M be a contact CR-submanifold of a quasi-Sasakian manifold $\tilde{M}$. Then we have:*

i) *The distribution $D \oplus \{\xi\}$ is integrable and its leaves are totally geodesic immersed in M if and only if*

$$(1.9) \quad h(X, Y) \in \Gamma(\mu), \quad \forall X \in \Gamma(D), \quad Y \in \Gamma(D \oplus \{\xi\}).$$

ii) *The distribution $D \oplus \{\xi\}$ is integrable and its leaves are totally geodesic immersed in $\tilde{M}$ if and only if*

$$(1.10) \quad h(X, Y) \in \Gamma(0) \quad \forall X \in \Gamma(D), \quad Y \in \Gamma(D \oplus \{\xi\}).$$

Theorem 1.3. Let M be a contact CR-submanifold of a quasi-Sasakian manifold $\tilde{M}$. Then we have:

i) The distribution $D^\perp$ is integrable and its leaves are totally geodesic immersed in M if and only if

$$(1.11) \quad h(X, Z) \in \Gamma(\mu), \quad \forall X \in \Gamma(D), \quad Z \in \Gamma(D^\perp),$$

and

$$(1.12) \quad FD^\perp \perp fD^\perp.$$

ii) The distribution $D^\perp$ is integrable and its leaves are totally geodesic in $\tilde{M}$ if and only if (1.12) is satisfied and

$$(1.13) \quad h(X, Z) \in \Gamma(\mu), \quad \forall X \in \Gamma(D \oplus D^\perp), \quad Z \in \Gamma(D^\perp).$$

Theorem 1.4. Let M be a contact CR-submanifold of a quasi-Sasakian manifold $\tilde{M}$. Then we have:

i) The leaves of distribution $D^\perp \oplus \{\xi\}$ are totally geodesic immersed in M if and only if

$$h(X, Z) \in \Gamma(\mu), \quad \forall X \in \Gamma(D), \quad Z \in \Gamma(D^\perp \oplus \{\xi\}).$$

ii) The leaves of distribution $D^\perp \oplus \{\xi\}$ are totally geodesic immersed in $\tilde{M}$ if and only if

$$h(X, Z) \in \Gamma(\mu), \quad \forall X \in \Gamma(D), \quad Z \in \Gamma(D^\perp \oplus \{\xi\})$$

and

$$h(Z, W) = 0, \quad \forall Z \in \Gamma(D^\perp \oplus \{\xi\}) \text{ and } W \in \Gamma(D^\perp).$$

2. CR-product of Quasi-Sasakian Manifold. The purpose of this paragraph is to study the existence of CR-products on quasi-Sasakian manifolds.

First we state

Theorem 2.1. Let M be a contact CR-submanifold of a quasi-Sasakian manifold $\tilde{M}$. Then the following assertions are equivalent:

i) M is $(D, D^\perp \oplus \{\xi\})$ contact CR-product;

ii) $D \perp FD$ and $\nabla_U Z \in \Gamma(D^\perp \oplus \{\xi\})$, $\forall U \in \Gamma(TM)$, $Z \in \Gamma(D^\perp)$;

iii) $D \perp FD$ and $Bh(X, Y) = 0$, $\forall X \in \Gamma(D)$, $Y \in \Gamma(TM)$;

iv) $D \perp FD$ and $h(fX, Y) = Ch(X, Y)$, $\forall X \in \Gamma(D)$, $Y \in \Gamma(TM)$;

v) $D \perp FD$ and $A_{fZ}X = 0$, $\forall X \in \Gamma(D)$, $Z \in \Gamma(D^\perp)$.

Proof. First we prove that assertions i) and ii) are equivalent, i) $\rightarrow$ ii).

Suppose that M is $(D, D^\perp \oplus \{\xi\})$ contact CR-product, and by using Theorems 1.1 and 1.4 it follows $D \perp FD$. Next by straight computation we infer

$$(2.1) \quad g(\nabla_X Z, fY) = g((\nabla_X f)Z - \nabla_X fZ, Y) = g(h(X, Y), fZ) = 0,$$

$$(2.2) \quad g(\nabla_W Z, fY) = -g(\tilde{\nabla}_W fZ, Y) = g(h((Y, W), fZ)) = 0,$$

for any $X, Y \in \Gamma(D)$, $Z \in \Gamma(D^\perp)$, $W \in \Gamma(D^\perp \oplus \{\xi\})$. From (2.1) and (2.2) it

follows assertion (ii).

ii) $\rightarrow$ i). The distribution $D^\perp \oplus \{\xi\}$ is always integrable, and let $X \in \Gamma(D)$, $Z \in \Gamma(D^\perp \oplus \{\xi\})$, $W \in \Gamma(D^\perp)$ and by using (1.1), (1.2), (1.4) and (1.5) we deduce

$$(2.3) \quad g(h(X, Z), fW) = g(\tilde{\nabla}_Z X, fW) = g((\tilde{\nabla}_Z f)X - \tilde{\nabla}_Z fX, W) = g(fX, \nabla_Z W) = 0.$$

By using Theorem 1.4 and relation (2.3) it follows that the leaves of distribution $D^\perp \oplus \{\xi\}$ are totally geodesic in M . Finally, let $X, Y \in \Gamma(D)$, $Z \in \Gamma(D^\perp)$ and by using (1.1), (1.2), (1.4) and (1.5) we infer

$$g(h(X, Y), fZ) = g(\tilde{\nabla}_X Y, fZ) = g((\tilde{\nabla}_X f)Y - \tilde{\nabla}_X fY, Z) = g(fY, \nabla_X Z) = 0.$$

By using (1.4) and Theorem 1.1 it follows that the distribution D is integrable and its leaves are totally geodesic in M .

Next we want to prove that assertions i) and iii) are equivalent, i) $\rightarrow$ iii). Suppose that M is $(D, D^\perp \oplus \{\xi\})$ contact CR-product. Then by using Theorems 1.1 and 1.4 it follows that $D \perp FD$ and

$$(2.4) \quad h(X, U) \in \Gamma(\mu), \quad \forall X \in \Gamma(D), \quad U \in \Gamma(TM).$$

But $\Gamma(\mu)$ is invariant by action of f and from (2.4) it follows that $Bh(X, U) = 0$, $\forall X \in \Gamma(D)$, $U \in \Gamma(TM)$ which prove assertion iii).

iii) $\rightarrow$ i). By using (1.3) and assertion iii) it follows that

$$fh(X, U) = Ch(X, U), \quad \forall X \in \Gamma(D), \quad U \in \Gamma(TM),$$

which is equivalent with

$$(2.5) \quad h(X, U) \in \Gamma(\mu), \quad \forall X \in \Gamma(D), \quad U \in \Gamma(TM).$$

Again by using Theorems 1.1 and 1.4 and relation (2.5) it follows assertion i).

Finally, we prove that assertions iv) and iii) are equivalent, iv) $\rightarrow$ iii).

Because $Bh(X, U) = 0$, $\forall X \in \Gamma(D)$, $U \in \Gamma(TM)$, it is easy to see that $h(X, U) \in \Gamma(\mu)$, $\forall X \in \Gamma(D)$, $U \in \Gamma(TM)$. Now let $V \in \Gamma(\mu)$ and by using (1.1), (1.2), (1.4) and (1.5) it follows

$$g(h(fX, U) - fh(X, U), V) = g((\tilde{\nabla}_U f)X, V) = 0,$$

which prove assertion iii).

iii) $\rightarrow$ iv). From $h(fX, U) = Ch(X, U)$ it follows that $h(fX, U) \in \Gamma(\mu)$ for any $\forall X \in \Gamma(D)$, $U \in \Gamma(TM)$, which prove assertion iv). The equivalence of v) and iv) follows from

$$g(A_f Z X, U) = g(h(X, U), fZ), \quad \forall X \in \Gamma(D), \quad U \in \Gamma(TM), \quad Z \in \Gamma(D^\perp).$$

Corollary 2.1. Let M be a contact CR-submanifold of a Sasakian manifold. Then M is not $(D, D^\perp \oplus \{\xi\})$ contact CR-product.

In the same way with Theorem 2.1 we can prove

Theorem 2.2. Let M be a contact CR-submanifold of a quasi-Sasakian manifold $\tilde{M}$. Then the following assertions are equivalent:

- i) M is $(D, \oplus\{\xi\}, D^\perp)$ contact CR-product;
- ii) $fD^\perp \perp FD^\perp$ and $\nabla_U X \in \Gamma(D \oplus \{\xi\})$, $\forall U \in \Gamma(TM)$, $X \in \Gamma(D)$;
- iii) $fD^\perp \perp FD^\perp$ and $Bh(X, U) = 0$, $\forall X \in \Gamma(D)$, $U \in \Gamma(TM)$;
- iv) $fD^\perp \perp FD^\perp$ and $h(fX, U) = Ch(X, U)$, $\forall X \in \Gamma(D)$, $U \in \Gamma(TM)$;
- v) $fD^\perp \perp FD^\perp$ and $A_{fZ}X = 0$, $\forall X \in \Gamma(D)$, $Z \in \Gamma(D^\perp)$.

Let M be a contact CR-submanifold of a quasi-Sasakian manifold $\tilde{M}$. Then we say that M is totally contact umbilical if then exists a normal vector field H so that the second fundamental form of M is given by

$$(2.6) \quad h(X, Y) = g(fX, fY) + \eta(X)h(Y, \xi) + \eta(Y)h(X, \xi), \quad \forall X, Y \in \Gamma(TM).$$

If we have

$$(2.7) \quad h(X, Y) = \eta(X)h(Y, \xi) + \eta(Y)h(X, \xi), \quad \forall X, Y \in \Gamma(TM),$$

then we say that M is totally contact geodesic.

Now we need the following (see [6])

L e m m a 2.1. *Let M be a contact CR-submanifold of a quasi-Sasakian manifold $\tilde{M}$. Then we have*

$$A_{fX} = A_{fY}X, \quad \forall X, Y \in \Gamma(D^\perp).$$

Next we prove

T h e o r e m 2.3. *Let M be a proper totally contact-umbilical of a quasi-Sasakian manifold $\tilde{M}$ with $\dim D^\perp > 1$. Then M is totally contact geodesic.*

P r o o f. From Lemma 2.1 we deduce

$$(2.8) \quad A_{fX}BH = A_{fBH}X, \quad \forall X \in \Gamma(D^\perp).$$

Because M is totally contact umbilical, from (2.8) we obtain

$$(2.9) \quad g(X, X)g(BH, BH) = g(X, BH)^2, \quad \forall X \in \Gamma(D^\perp).$$

But $\dim D^\perp > 1$ and from (2.9) it follows that $BH = 0$.

Now, by using formula (from [6])

$$(\nabla_Z B)N = A_{CN}Z - t(A_N Z) + g(FZ, N)\xi, \quad \forall Z \in \Gamma(TM), \quad N \in \Gamma(TM^\perp),$$

we obtain

$$(2.10) \quad PA_{CN}Z = tA_N Z, \quad \forall Z \in \Gamma(TM), \quad N \in \Gamma(TM^\perp).$$

We have further,

$$(2.11) \quad g(PA_{fH}Z, Y) = g(A_{fH}Z, Y) = g(Y, Z)g(H, fH) = 0,$$

$$\forall Y \in \Gamma(D), \quad Z \in \Gamma(TM)$$

and

$$(2.12) \quad g(tA_{fH}Z, Y) = -g(A_{fH}Z, fY) = -g(fY, Z)g(H, H).$$

From (2.10), (2.11) and (2.12) we obtain $H = 0$.

T h e o r e m 2.4. *Let M be a proper totally contact geodesic submanifold of a quasi-Sasakian manifold $\tilde{M}$. If $FD \subseteq \Gamma(TM)$, then M is*

locally a $(D \oplus \{\xi\}, D^\perp)$ CR-product $M_1 \times M_2$, where M_1 is a totally geodesic invariant submanifold of $\tilde{M}$, and M_2 is totally geodesic anti-invariant submanifold of $\tilde{M}$.

P r o o f. From (2.7) we deduce $h(X, Y) = 0, \forall X, Y \in \Gamma(D \oplus D^\perp)$ and by using (1.2) we infer

$$(2.13) \quad h(X, \xi) = \beta fX, \quad \forall X \in \Gamma(D).$$

Now from (2.13) and Theorem 2.1 we deduce our assertion.

From Theorem 2.4 we obtain

C o r o l l a r y 2.2. Let M be a proper totally contact umbilical submanifold of a quasi-Sasakian manifold $\tilde{M}$ so that $\Gamma(FD) \subseteq \Gamma(D^\perp)$. If $\dim D^\perp > 1$, then M is $(D, D^\perp \oplus \{\xi\})$ CR-product $M_1 \times M_2$, where M_1 is a totally geodesic invariant submanifold of $\tilde{M}$, and M_2 is totally geodesic anti-invariant submanifold of $\tilde{M}$.

Now from Theorem 2.2 and relation (2.13) we deduce

T h e o r e m 2.5. Let M be a proper totally contact geodesic submanifold of a quasi-Sasakian manifold $\tilde{M}$. If $\Gamma(FD) \subseteq \Gamma(TM)$ and $FD^\perp \perp fD^\perp$, then M is locally a $(D \oplus \{\xi\}, D^\perp)$ CR-product $M_1 \times M_2$, where M_1 is a totally geodesic invariant submanifold of $\tilde{M}$, and M_2 is totally geodesic anti-invariant submanifold of $\tilde{M}$.

C o r o l l a r y 2.3. Let M be a proper totally contact geodesic submanifold of a quasi-Sasakian manifold $\tilde{M}$, so that $\Gamma(FD) \subseteq \Gamma(TM)$ and $FD^\perp \perp fD^\perp$, then M is locally a $(D \oplus \{\xi\}, D^\perp)$ CR-product $M_1 \times M_2$, where M_1 is a totally geodesic invariant submanifold of $\tilde{M}$, and M_2 is totally geodesic anti-invariant submanifold of $\tilde{M}$.

Received November 20, 1992

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CR-PRODUSE ALE UNEI VARIETĂȚI QUASI-SASAKI

(Rezumat)

Se stabilesc condițiile necesare și suficiente pentru ca o CR-subvarietate a unei varietăți quasi-Sasaki să fie CR-produs.

ON TWO INTEGRAL INEQUALITIES

BY

IOSIP E. PEČARIĆ

1. The following result is given in the books [1; II, pp. 57, 214] and [2, p. 166]:

If $a \geq 0$, $b \geq 0$, $a \neq 1$, and ϕ is nonnegative and increasing, then

$$(1) \quad \left[\int_0^1 x^{a+b} \phi(x) dx \right]^2 > \left[1 - \left(\frac{a-b}{a+b+1} \right)^2 \right] \int_0^1 x^{2a} \phi(x) dx \int_0^1 x^{2b} \phi(x) dx,$$

unless $\phi(x) = \text{const.}$

A. M. Fink and Max Jodeit Jr. [3] showed that (1) is valid if $a, b > -1/2$ even if $a=1$. In fact, they proved that (1) is equivalent to

$$\int_0^1 \int_0^1 \phi(x) \phi(y) K(x, y) dx dy \geq 0,$$

where

$$K(x, y) = (a+b+1)^2 x^{a+b} y^{a+b} - \frac{1}{2} (2a+1)(2b+1) (x^{2a} y^{2b} + x^{2b} y^{2a}).$$

If $G(s, t) = \int_s^1 \int_t^1 K(x, y) dx dy$ then (see [3]) $G(s, t) \geq 0$ on $[0, 1]^2$ and for $a \neq b$,

$G(s, t) = 0$ only if $(s, t) = (0, 0)$ or $s=1$ or $t=1$.

First, we shall note that the following identity is valid (see [4]):

$$\int_c^d \int_c^d p(x,y) f(x) g(y) dx dy = f(c) g(c) P(c,c) + g(c) \int_c^d P(x,c) df(x) +$$

$$+ f(c) \int_c^d P(c,y) dg(y) + \int_c^d \int_c^d P(x,y) df(x) dg(y),$$

where $P(x,y) = \int_x^d \int_y^d p(s,t) ds dt$.

By using this result in the case $c=0$, $d=1$, $p(x,y)=K(x,y)$, we have

Theorem 1. If $a, b > -1/2$, and $f, g: [0, 1] \rightarrow \mathbb{R}$ are nonnegative and nondecreasing, then

$$(2) \quad 2(a+b+1) \int_0^1 x^{a+b} f(x) dx \int_0^1 x^{a+b} g(x) dx \geq (2a+1)(2b+1) \cdot$$

$$\cdot \left[\int_0^1 x^{2a} f(x) dx \int_0^1 x^{2b} g(x) dx + \int_0^1 x^{2b} f(x) dx \int_0^1 x^{2a} g(x) dx \right].$$

Moreover, the following identity is given in [5]:

$$\int_a^b \int_a^b p(x,y) f(x,y) dx dy = f(a,a) P(a,a) + \int_a^b P(x,a) f_1(x,a) dx +$$

$$+ \int_a^b P(a,y) f_2(a,y) dy + \int_a^b \int_a^b P(x,y) f_{21}(x,y) dx dy,$$

where P is defined as in above and f_1, f_2 and f_{21} are partial derivatives $\partial f / \partial x$, $\partial f / \partial y$ and $\partial^2 f / \partial x \partial y$ respectively.

So, we can prove the following

Theorem 2. If $f(0,0)$, $f_1(x,0)$, $f_2(0,y) \geq 0$ and $f_{21}(x,y) \geq 0$ on $[0,1] \times [0,1]$, and $a, b > -1/2$, then

$$(3) \quad 2(a+b+1)^2 \int_0^1 \int_0^1 f(x,y) x^{a+b} y^{a+b} dx dy \geq$$

$$\geq (2a+1)(2b+1) \int_0^1 \int_0^1 f(x,y) (x^{2a} y^{2b} + x^{2b} y^{2a}) dx dy.$$

Remark. Under some assumptions about the functions f and g the inequality (2) can be deduced easily from (3).

2. A function $f: I \times I \rightarrow [0, \infty)$ is said to be a positive set function (P. S. F.) if

(4) $\Delta_{hk}^2 f(s, t) := f(s+h, t+k) - f(s+h, t) - f(s, t+k) + f(s, t) \geq 0$
 holds for all $h, k \geq 0$ (or $h, k \leq 0$) with $s+h, t+k \in I$ and arbitrary $s, t \in I$. Of course, if f is smooth, i.e. f has continuous second partial derivatives, then (4) is equivalent to $f_{21}(s, t) \geq 0$.

Using a method from [5], H. P. Heinig and L. Maligranda [6] proved

Theorem A. Let K be a positive integrable function on $I \times I$ and suppose $f: I \times I \rightarrow [0, \infty)$ is a continuous P.S.F. ($I = [0, a]$).

a) If for all $t \in I$

$$(5) \quad \int_t^a \int_0^t K(u, v) dv du = \int_0^t \int_t^a K(u, v) dv du,$$

then

$$(6) \quad \int_0^a \int_0^a K(s, t) f(s, t) ds dt \leq \int_0^a \left[\int_0^a K(s, t) ds \right] f(t, t) dt.$$

b) If for all $s, t \in I$

$$(7) \quad \int_0^a K(u, t) du = B w(t) \quad \text{and} \quad \int_0^a K(s, v) dv = B w(s),$$

then

$$(8) \quad \int_0^a \int_0^a K(s, t) f(s, t) ds dt \leq B \int_0^a w(s) f(s, s) ds.$$

Here we shall show that this result of H. P. Heinig and L. Maligranda can be obtained from a result given in [5].

We shall consider the following expressions:

$$U(f, K) = \int_0^a \left[\int_0^a K(s, t) ds \right] f(t, t) dt - \int_0^a \int_0^a K(s, t) f(s, t) ds dt,$$

$$V(f, k) = B \int_0^a w(s) f(s, s) ds - \int_0^a \int_0^a K(s, t) f(s, t) ds dt,$$

where f, K are integrable functions, K is positive and B and w satisfy (7).

Precisely, we shall prove the following

Theorem 3. Let K be a positive integrable function on $I \times I$ and suppose that $f: I \times I \rightarrow \mathbb{R}$ has the continuous partial derivatives f_1, f_2 and f_{21} on $I \times I$.

a) If for all $t \in I$, (4) is valid, then

$$(9) \quad U(f, K) = F_{21}(\xi, \eta) U(xy, K), \quad (\xi, \eta) \in I.$$

b) If for all $s, t \in I$, (6) is valid, then

$$(10) \quad V(f, K) = f_{21}(\alpha, \beta) V(xy, K), \quad (\alpha, \beta \in I).$$

P r o o f. We shall show that this result is a consequence of Theorem 15 from [5]:

L e m m a. Let $p: I \times I \rightarrow \mathbb{R}$ and $q: I \rightarrow \mathbb{R}$ be integrable functions such that

$$(11) \quad P(0, y) = Q(y), \quad P(x, 0) = Q(x),$$

and

$$(12) \quad P(x, y) \leq Q(\max(x, y))$$

$$(\forall x, y \in I), \text{ where } Q(x) = \int_x^a q(t) dt, \quad P(x, y) = \int_x^a \int_y^a p(s, t) dt ds.$$

If $f: I \times I \rightarrow \mathbb{R}$ has the continuous partial derivatives f_1, f_2 and f_{21} on $I \times I$, then

$$(13) \quad F(f, p, q) = f_{21}(\xi, \eta) F(xy, p, q), \quad (\xi, \eta \in I),$$

where

$$F(f, p, q) = \int_0^a q(x) f(x, x) dx - \int_0^a \int_0^a p(x, y) f(x, y) dx dy.$$

For a p r o o f of a) we should set: $p(s, t) \rightarrow K(s, t)$, $q(t) = \int_0^a K(s, t) ds$.

Then we have $F(f, p, q) = U(f, K)$. Further,

$$P(0, y) = \int_0^a \int_y^a K(s, t) ds dt = \int_y^a \left[\int_0^a K(s, t) ds \right] dt = Q(y),$$

$$\begin{aligned} P(x, 0) &= \int_x^a \int_0^a K(s, t) ds dt = \int_x^a \int_0^x K(s, t) ds dt + \int_x^a \int_x^a K(s, t) ds dt = \\ &= \int_0^x \int_x^a K(s, t) ds dt + \int_x^a \int_x^a K(s, t) ds dt = \int_0^x \int_x^a K(s, t) ds dt = \\ &= \int_x^a \left[\int_0^a K(s, t) ds \right] dt = Q(x). \end{aligned}$$

Thus, (11) is valid.

Now, we shall show that (12) is valid too. For $y > x$ we have

$$\begin{aligned}
 P(x, y) &= \int_x^a \int_y^a K(s, t) ds dt \leq \int_0^a \int_y^a K(s, t) ds dt = \\
 &= \int_y^a \left[\int_0^a K(s, t) ds \right] dt = Q(y) = Q(\max(x, y)).
 \end{aligned}$$

For $x > y$ we have

$$\begin{aligned}
 P(x, y) &= \int_x^a \int_y^a K(s, t) ds dt \leq \int_x^a \int_0^a K(s, t) ds dt = \int_x^a \int_0^x K(s, t) ds dt + \\
 &+ \int_x^a \int_x^a K(s, t) ds dt = \int_0^x \int_x^a K(s, t) ds dt + \int_x^a \int_x^a K(s, t) ds dt = \\
 &= \int_0^x \int_0^a K(s, t) ds dt = \int_x^a \left[\int_0^a K(s, t) ds \right] dt = Q(x) = Q(\max(x, y)).
 \end{aligned}$$

Thus, (12) is valid too. Therefore, from (13) we get (9).

For a proof of b) we should take $p(s, t) = K(s, t)$, $q(s) = Bw(s)$. Then we have $F(f, p, q) = V(f, K)$, and

$$\begin{aligned}
 P(0, y) &= \int_0^a \int_y^a K(s, t) ds dt = \int_y^a \left[\int_0^a K(s, t) ds \right] dt = \int_x^a Bw(t) dt = Q(y), \\
 P(x, 0) &= \int_x^a \int_0^a K(s, t) ds dt = \int_x^a \left[\int_0^a K(s, t) dt \right] ds = \int_x^a Bw(s) ds = Q(x),
 \end{aligned}$$

and for $x > y$,

$$P(x, y) = \int_x^a \int_y^a K(s, t) ds dt \leq \int_x^a \left[\int_0^a K(s, t) dt \right] ds = \int_x^a Bw(s) ds = Q(x) = Q(\max(x, y))$$

or for $y > x$,

$$P(x, y) = \int_x^a \int_y^a K(s, t) ds dt \leq \int_y^a \left[\int_0^a K(s, t) ds \right] dt = \int_y^a Bw(t) dt = Q(y) = Q(\max(x, y)).$$

Therefore, the conditions (11) and (12) are fulfilled, and from (13) we get (10). Thus, we just proved our theorem.

Using this result and the same approximation procedure as in [5] and [6], we can obtain the result of Heinig and Maligranda.

Moreover, using Theorem 16 from [5] and the method of proof of Theorem 3, we can similarly prove the following

Theorem 4. Let K be a positive integrable function on $I \times I$ and suppose that $f, g: I \times I \rightarrow \mathbb{R}$ have continuous partial derivatives $f_1, f_2, f_{21}, g_1, g_2$ and $g_{21} \neq 0$ on $I \times I$.

a) If for all $t \in I$, (4) is valid, then

$$U(f, K) = \frac{f_{21}(\xi, \eta)}{g_{21}(\xi, \eta)} U(g, K), \quad (\xi, \eta \in I).$$

b) If for all $s, t \in I$, (6) is valid, then

$$V(f, K) = \frac{f_{21}(\alpha, \beta)}{g_{21}(\alpha, \beta)} V(g, K), \quad (\alpha, \beta \in I).$$

Received February 10, 1994

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ASUPRA A DOUĂ INEGALITĂȚI INTEGRALE

(Rezumat)

Se stabilesc noi inegalități integrale, care se adaugă la altele de dată recentă, consemnate de literatura de specialitate [3]...[6].

THE DIFFERENTIAL SYSTEM $Ay' + By = 0$

BY

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The starting point of this Note is the article [1] by the regretted Professor Al. C. C l i m e s c u, who honoured for almost half a century the Department of Mathematics of the Polytechnic Institute of Jassy. He stated a result on the general solution of the linear differential system

$$(1) \quad Ay' + By = 0,$$

where $A \neq 0$ and B are constant matrices of type $n \times n$, $y = y(t)$ is the unknown vector-function and $y' = y'(t)$ stands for its derivative with respect to the independent variable t . Of course, it is assumed that $n \geq 2$. It is also assumed that, for each j ($1 \leq j \leq n$), the columns of rank j of matrices A and B are not both null. This condition is required to ensure that the number of variables (unknown functions) in Eq. (1) is exactly equal to n . Our aim is to present a method for solving system (1), which indicates (in each case) the number of arbitrary constants and arbitrary functions involved in its general solution. We search for the solutions of system (1) in the class of the continuous differentiable functions.

(i) *Case when $\det A \neq 0$.* This is the simplest possible case since we can write

$$(2) \quad y' = Cy, \quad C = -A^{-1}B,$$

where C is a constant square matrix of order n . The general solution of (2) is given by $y = Y(t)c$, where $Y(t) = \exp(Ct)$ and c is an arbitrary constant vector of type $n \times 1$. Hence the general solution contains n arbitrary constants $c_i, \overline{1, n}$ — the entries of vector c .

(ii) *Case when $\det A=0$, $\det B \neq 0$.* We can now write the system (1) under the equivalent form

$$(3) \quad y = Cy', \quad C = -B^{-1}A,$$

where C is a constant matrix with $\det C=0$. If the matrix C is nilpotent, i.e., there exists an integer number m ($1 \leq m \leq n$) such that $C^m=0$ = the zero matrix of order n , then we have the nice result [1, Theorem II]: *The necessary and sufficient condition for the system (1) to admit the null solution only is that it may be written under the form (3), where C is a nilpotent matrix.*

We remark that in a given (real) case, it is not necessary to verify whether C is nilpotent or not. Indeed, let us assume that $\text{rank } C=p$ ($1 \leq p \leq n-1$). Without any loss of generality, we may suppose that the first p rows of C are linearly independent and therefore the remaining $n-p$ rows are linear combinations of them. Then, we deduce that the last $n-p$ variables are linear combinations of the first p variables, i.e., we have

$$(4) \quad y_k = \sum_{j=1}^p \alpha_{kj} y_j, \quad k = \overline{p+1, n},$$

where α_{kj} are constants. Using (4), we obtain from (3) a new system of the form

$$(5) \quad y_i = \sum_{j=1}^p d_{ij} y_j', \quad i = \overline{1, p},$$

where d_{ij} are constants. If we use the notations $\tilde{y}$ and D given by

$$(6) \quad \tilde{y}^T = [y_1 y_2 \dots y_p] \quad \text{and} \quad D = [d_{ij}],$$

the system (5) may be written as $\tilde{y} = D\tilde{y}'$, entirely analogous to system (3). Simple examples show that both the cases $\det D \neq 0$ or $\det D = 0$ are possible.

Example. For the following system (with $n=3$, $p=2$)

$$(7) \quad \begin{aligned} y_1 &= y_1' + y_2' + y_3', \\ y_2 &= 2y_1' - y_2' + y_3', \\ y_3 &= y_1' - 2y_2', \end{aligned}$$

we have $y_3 = -y_1 + y_2$ and $\det D \neq 0$, while for the system (with $n=3$, $p=2$ too) obtained from (7) by replacing the third equation by $y_3 = -6y_1' + 3y_2' - 3y_3'$, we have $y_3 = -3y_2$ and $\det D = 0$.

Let us now return to the system (5). If $\det D = 0$, it may be written in the normal form

$$(8) \quad y' = M\tilde{y}, \quad M = D^{-1} = \text{a constant matrix},$$

and its general solution is given by $\tilde{y} = \exp(Mt)\tilde{c}$, where $\tilde{c}$ is an arbitrary column-vector of type $p \times 1$. In this case, using Eqs. (4) again, we conclude that the general solution of the system (3) contains exactly p arbitrary constants (the

entries of the vector $\tilde{c}$) and it does not contain arbitrary functions. If $\det D=0$, the system $\tilde{y}'=D\tilde{y}'$ is of the same type as the initial system (3), with the single difference that the number of variables is now $p < n$. We must repeat the above described procedure, eliminating a part of the variables $y_1, y_2, \dots, y_p$.

The case when all the variables except one, say y_1 , are eliminated is not excluded, finally obtaining a single equation $y_1' = ay_1$, $a = \text{const}$. If $a \neq 0$, we have $y_1 = c_1 \exp(bt)$ with $b = a^{-1}$, the general solution of (3) containing a single arbitrary constant c_1 . If $a = 0$, we deduce that the system (3) has only the trivial solution $y_i = 0$, $i = \overline{1, n}$, and we know that this is the case when C is nilpotent. We can summarize the above discussion stating

Proposition 1. Assume that, in system (3), $\text{rank } C = p$ ($1 \leq p < n$). Then, by (possible successive) elimination of a part of variables, we can obtain:

- i) a system in the normal form, consisting of p equations. The general solution of (3) will contain exactly p arbitrary constants;
- ii) a system in the normal form, consisting of k equations ($1 \leq k < p$). The general solution of (3) will contain exactly k arbitrary constants;
- iii) only the null solution $y_i = 0$, $i = \overline{1, n}$.

A similar Proposition may be formulated concerning the nonhomogeneous system corresponding to (3), namely

$$(9) \quad y' = Cy + f(t),$$

where C is a constant matrix of order n , $f = f(t)$ is a given vector-function whose entries $f_i = f_i(t)$, $i = \overline{1, n}$ have derivatives of sufficiently large order and $y = y(t)$ is the unknown vector-function. We only remark that if C is nilpotent, the system (9) has a unique solution.

(iii) Case when $\det A = \det B = 0$. Now, the system (1) cannot be written under any of the forms (2) or (3), and we shall use another algorithm to solve it.

Let us assume that $\text{rank } A = p$ ($1 \leq p < n$). Since it is not restrictive to suppose the first p rows of A as being linearly independent, we obtain after some elementary transformations the system (1) reduced to one of the equivalent forms

$$(10) \quad \sum_{j=1}^n a_{ij} y_j' + \sum_{j=1}^n b_{ij} y_j = 0, \quad i = \overline{1, p},$$

or

$$(11) \quad \sum_{j=1}^n a_{ij} y_j' + \sum_{j=1}^n b_{ij} y_j = 0, \quad i = \overline{1, p},$$

$$\sum_{j=1}^n d_{kj} y_j = 0, \quad k = \overline{1, q},$$

where it is assumed that $\text{rank}[d_{kj}] = q \geq 1$, $p + q \leq n$.

(iii.1) If the system (1) can be reduced to the form (10) and if we suppose that the first p columns of A are linearly independent, we may take $y_{p+1}, y_{p+2}, \dots, y_n$ as arbitrary continuously differentiable functions. Then, to determine the first p unknown functions $y_1, y_2, \dots, y_p$ we have to solve a normal differential system. Hence, in this case, the general solution of (1) contains p arbitrary constants and $n - p$ arbitrary continuously differentiable functions.

If the system (1) can be reduced to the form (11), we may eliminate q variables, say the last q ones, from the linear homogeneous algebraic system consisting of the last q equations in (11). In other words, we may express $y_{n-q+1}, y_{n-q+2}, \dots, y_n$ as linear combinations of $y_1, y_2, \dots, y_{n-q}$. The situation when these last q variables (or a part of them) are identically zero is not excluded. Then, from the first p equations of (11), we obtain a system of the form

$$(12) \quad \sum_{j=1}^{n-q} g_{ij} y_j' + \sum_{j=1}^{n-q} h_{ij} y_j = 0, \quad i = \overline{1, p},$$

where g_{ij} and h_{ij} are constants. Here, we have p equations in $n - q$ unknown functions $y_1, y_2, \dots, y_{n-q}$ ($1 \leq p \leq n - q$). Various situations may appear in this subcase, namely:

(iii.2) $p = n - q$, $\det [g_{ij}] \neq 0$. The system (12) may be written in the normal form

$$(13) \quad y_i' = \sum_{j=1}^p m_{ij} y_j, \quad i = \overline{1, p},$$

where m_{ij} are constants. The general solution of (1) will contain exactly p arbitrary constants but it will contain no arbitrary functions.

(iii.3) $p = n - q$, $\det [g_{ij}] = 0$, $\det [h_{ij}] \neq 0$. The system (12) may now be written as

$$(14) \quad y_i = \sum_{j=1}^p r_{ij} y_j', \quad i = \overline{1, p},$$

where r_{ij} are constants and $\det [r_{ij}] = 0$. Making use of Proposition 1, we conclude that in this subcase the general solution of (1) will contain k arbitrary constants ($1 \leq k < p$) or this system will admit the null solution only. We also remark that nor in this subcase the solution contains any arbitrary functions.

(iii.4) $p = n - q$, $[g_{ij}] = \theta =$ the zero matrix of order p , and $\det [h_{ij}] = 0$. There exist very simple examples which prove that this subcase is possible, but we omit the details. This subcase is interesting because the general solution of (1) now contains only continuously differentiable functions (whose number is $\leq p$) but it does not explicitly contain arbitrary constants.

(III.5) $p = n - q$, $[g_{ij}] \neq 0$, $\det [g_{ij}] = \det [h_{ij}] = 0$. Now, we are exactly in the same situation as in the case (iii) for system (1) and we must repeat the above elimination procedure. The advantage is that the number of equations (and of variables) is now less than in the initial case ($p < n$). After a finite number of steps, we are in the position to write down the solution of (1), if we also take into account the subcases presented below.

(iii.6) $p < n - q$, $\text{rank } [g_{ij}] = p$. In this subcase, we can take $n - q - p$ variables, say $y_{p+1}, y_{p+2}, \dots, y_{n-q}$ as arbitrary continuously differentiable functions. Then, we obtain from (12) a normal linear differential system in variables $y_1, y_2, \dots, y_p$. The general solution of (1) will contain p arbitrary constants and $n - q - p$ arbitrary continuously differentiable functions.

(iii.7) $p < n - q$, $[g_{ij}] = 0$ = the null matrix of size $p \times (n - q)$. To prove that this subcase is possible, it suffices to suppose that (11) with $n=3$, $p=1$, $q=1$ is given by

$$(15) \quad y_1' + y_1 + y_2 + y_3 = 0, \quad y_1 = 0.$$

Then, the system (12) reduces to a single equation, namely $y_2 + y_3 = 0$. Hence, the general solution of the system (1), from which (15) is derived, is given by $y_1 = 0$, $y_2 = \varphi$, $y_3 = -\varphi$, where φ is an arbitrary function. This solution does not contain arbitrary constants. Coming back to the general case described by (iii.7), the system (12) becomes linear algebraic, with p equations and $n - q$ unknown functions:

$$(16) \quad \sum_{j=1}^{n-q} h_{ij} y_j = 0, \quad i = \overline{1, p}.$$

Here, we again encounter the situation when the general solution of (1) does not explicitly contain arbitrary constants, but contains m arbitrary functions, with $1 \leq m \leq n - q$.

(iii.8) $p < n - q$, $\text{rank } [g_{ij}] = k$ ($1 \leq k < p$). Starting from the system (12), we have to repeat the elimination procedure thus obtaining a new system of the form (10) or (11). If we arrive at a system of form (10), it will consist of k equations in which the unknown functions $y_1, y_2, \dots, y_{n-q}$ and their first derivatives appear. The advantage is that the number of equations and of variables is now smaller than for the system (10) because we have $k < p$ and $n - q < n$. If we arrive to a system of form (11) it will contain k equations in terms of the unknown functions $y_1, y_2, \dots, y_{n-q}$ and of their first derivatives, as well as a number of r linear homogeneous (linearly independent) algebraic equations in the same variables. Of course, we have $k + r \leq p$.

Continuing this way, that is, repeating (if necessary) the elimination procedure for a part of the remaining variables, it is clear that we shall arrive—

after a finite number of steps — either to a differential system in the normal form or to an algebraic linear system; a part of the unknowns of the latter system can be taken as arbitrary continuously differentiable functions (in subcase (iii.2) such "free" variables do not exist). It is also possible that this final system consists of a single equation only, in which enter a part of the remaining variables and their first derivatives. We consider an example showing that it is possible for this "final equation" to be an identity or, in other words, to have all its coefficients equal to zero.

Example. For the following system of form (11) (with $n=3$, $p=1$, $q=1$), namely

$$(17) \quad y_1' - y_2' - y_3' = 0, \quad y_1 - y_2 - y_3 = 0,$$

we have $y_1 = y_2 + y_3$ after elimination of the first variable. Introducing it in the first equation of (17) we obtain an identity and therefore y_2, y_3 are free variables.

To summarize the above discussion on the case (iii), let us denote by μ the number of arbitrary constants and by ν the number of arbitrary functions which enter in the general solution of the system (1). We can now state the following

Proposition 2. Assume that, in (1), $\det A = \det B = 0$ and $\text{rank } A = p$ ($1 \leq p < n$). Then

1° if system (1) can be reduced to the form (10), its general solution will contain p arbitrary constants and $n - p$ arbitrary continuously differentiable functions, i.e., $\mu = p$ and $\nu = n - p$;

2° if system (1) can be reduced to the form (11), then we have $0 \leq \mu \leq p$, $0 \leq \nu \leq n - q$, $1 \leq \mu + \nu \leq n - q$.

The conclusion that $\mu = \nu = 0$ is not possible also follows from the above discussion. But we also can justify it otherwise: the hypothesis $\mu = \nu = 0$ implies that system (1) with $\det A = \det B = 0$ has a unique solution (or a finite number of solutions) which, obviously, is a contradiction.

Propositions 1 and 2 of this paper are closely related with Theorem I, which is the main result of [1]. The method used here may also be applied to linear systems of the form

$$(18) \quad \sum_{j=1}^n a_{ij} y_j' + \sum_{j=1}^n b_{ij} y_j = 0, \quad i = \overline{1, p},$$

with $p < n$, where $A = [a_{ij}] \neq 0$ = the zero matrix of size $p \times n$.

Received October 17, 1992

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SISTEMUL DIFERENȚIAL $Ay' + By = 0$

(Rezumat)

Se demonstrează două Propoziții despre soluția generală a acestui sistem în cazurile:
(1) $\det A = 0$, $\det B \neq 0$ și respectiv (2) $\det A = \det B = 0$.

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RECEIVED AT THE EDITORIAL OFFICE

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JORDAN BASES AND DIFFERENTIAL LINEAR SYSTEMS WITH CONSTANT COEFFICIENTS

BY

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1. Introduction. Consider the linear space

$$C_n = \left\{ x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, x_j \in \mathbb{C}, j = \overline{1, n} \right\}$$

and the matrix $A = (a_{jk})_{n \times n}$, $a_{jk} \in \mathbb{R}$, with its characteristic polynomial

$$(1) \quad p(\lambda) = |\lambda E_n - A|, \quad \lambda \in \mathbb{C}, \quad E_n \text{ the unit matrix.}$$

If

$$z = \begin{bmatrix} x_1 + iy_1 \\ \vdots \\ x_n + iy_n \end{bmatrix}, \quad x_j, y_j \in \mathbb{R}, \quad j = \overline{1, n},$$

is a vector of C_n , we denote

$$\Re z = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \Im z = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \quad \text{and} \quad \bar{z} = \begin{bmatrix} x_1 - iy_1 \\ \vdots \\ x_n - iy_n \end{bmatrix} = \Re z - i \Im z$$

($\bar{z}$ is the complex conjugate of $z = \Re z + i \Im z$).

A Jordan basis in $\mathbb{C}_n$ exists for the matrix A [6]...[8]. Starting from such a Jordan basis, we shall prove the existence of a Jordan basis of the form

$$(2) \quad \left\{ u_1^1, \dots, u_{m_1}^1, \dots, u_1^p, \dots, u_{m_p}^p, v_1^1, \dots, v_{n_1}^1, \dots, \right. \\ \left. \dots, v_1^r, \dots, v_{n_r}^r, w_1^1, \dots, w_{n_1}^1, \dots, w_1^r, \dots, w_{n_r}^r \right\},$$

where

$$(3) \quad \begin{aligned} Au_1^k &= \lambda_k u_1^k, \quad Au_2^k = \lambda_k u_2^k + u_1^k, \dots, \\ Au_{m_k}^k &= \lambda_k u_{m_k}^k + u_{m_k-1}^k, \quad k = \overline{1, p}; \\ Av_1^k &= \mu_k v_1^k, \quad Av_2^k = \mu_k v_2^k + v_1^k, \dots, \\ Av_{n_k}^k &= \mu_k v_{n_k}^k + v_{n_k-1}^k, \quad k = \overline{1, r}; \\ Aw_1^k &= \bar{\mu}_k w_1^k, \quad Aw_2^k = \bar{\mu}_k w_2^k + w_1^k, \dots, \\ Aw_{n_k}^k &= \bar{\mu}_k w_{n_k}^k + w_{n_k-1}^k, \quad k = \overline{1, r}, \end{aligned}$$

$\lambda_1, \dots, \lambda_p \in \mathbb{R}$, $\mu_1, \dots, \mu_r$ and the complex conjugate $\bar{\mu}_1, \dots, \bar{\mu}_r \in \mathbb{C}$, $u_j^k, k = \overline{1, p}$, $j = \overline{1, m_k}$, have the real components, $w_j^k = \bar{v}_j^k, k = \overline{1, r}, j = \overline{1, n_k}$.

The basis (2) will be used to find a fundamental system of solution with real components, for the differential linear system with constant coefficients

$$x'(t) = Ax(t), \quad t \in \mathbb{R}, \quad x(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}.$$

2. Some Properties of the Series of Eigen-Vectors and Associated Vectors. The Existence of a Basis (2). Consider the matrix $A = (a_{jk})_{n \times n}$, $a_{jk} \in \mathbb{R}$ and let λ_0 be a root of the characteristic polynomial (1).

A system of vectors $\{v_1, \dots, v_p\} \subset \mathbb{C}_n$, so that

$$(4) \quad v_1 \neq 0, \quad Av_1 = \lambda_0 v_1, \quad Av_j = \lambda_0 v_j + v_{j-1}, \quad j = \overline{2, p},$$

is called series of eigen-vectors and associated vectors, of the matrix A , corresponding to eigenvalue λ_0 .

A Jordan basis in $\mathbb{C}_n$, of the matrix A , is a basis in $\mathbb{C}_n$, consisting of series of eigen-vectors and associated vectors, of the matrix A .

Theorem 1. Let $\left\{ v_1^j, \dots, v_{n_j}^j \right\}, j = \overline{1, p}$, be a series of eigen-vectors

and associated vectors, corresponding to eigenvalues $\lambda_1, \dots, \lambda_p$ of the matrix A .
If

(H) $v_1^1, \dots, v_1^p$ are linearly independent,

then

(5) $v_1^1, \dots, v_{n_1}^1, \dots, v_1^q, \dots, v_{n_q}^q, v_1^{q+1}, \dots, v_1^p, 1 \leq q \leq p$
are linearly independent.

P r o o f. We shall prove by induction with respect to q . Assume that $q=1$. In this case the statement will be established by induction with respect to n_1 ; for $n_1=1$ the assertion is true by hypothesis (H). Suppose now that it is true for $n_1 - 1$ and we prove that it is true for n_1 .

Let $\alpha_1^1, \dots, \alpha_{n_1}^1, \alpha_1^2, \dots, \alpha_1^p \in \mathbb{C}$ be so that

$$(6) \quad \sum_{j=1}^{n_1} \alpha_j^1 v_j^1 + \sum_{k=2}^p \alpha_1^k v_1^k = 0.$$

From (6) it follows $A \left[\sum_{j=1}^{n_1} \alpha_j^1 v_j^1 + \sum_{k=2}^p \alpha_1^k v_1^k \right] = 0$, and then

$$(7) \quad \sum_{j=1}^{n_1} \lambda_1 \left(\alpha_j^1 v_j^1 \right) + \sum_{j=1}^{n_1-1} \alpha_{j+1}^1 v_j^1 + \sum_{k=2}^p \alpha_1^k \lambda_k v_1^k = 0.$$

Replacing in (7) $\sum_{j=1}^{n_1} \alpha_j^1 v_j^1$ of (6), it results

$$(8) \quad \sum_{j=1}^{n_1-1} \alpha_{j+1}^1 v_j^1 + \sum_{k=2}^p \alpha_1^k (\lambda_k - \lambda_1) v_1^k = 0.$$

Taking the induction hypothesis into account, from (8) it follows

$$(9) \quad \alpha_{j+1}^1 = 0, \quad j = \overline{1, n_1 - 1}.$$

Replacing (9) in (8) and taking into account the hypothesis (H), it follows also $\alpha_1^1 = 0, \alpha_1^k = 0, k = \overline{2, p}$.

Next, we suppose that the statement is true for $q - 1$, and we prove, by induction with respect to n_q , that it is true for q . For $n_q=1$ the statement is true of the induction hypothesis relative to q . Assume that it is true for $n_q - 1$ and we prove that it is true for n_q . Let $\alpha_k^j \in \mathbb{C}, k = \overline{1, n_j}, j = \overline{1, q}, \alpha_1^{q+1}, \dots, \alpha_1^p \in \mathbb{C}$ be so that

$$(10) \quad \sum_{j=1}^q \left[\sum_{k=1}^{n_j} \alpha_k^j v_k^j \right] + \sum_{j=q+1}^p \alpha_1^j v_1^j = 0.$$

From (10) it follows $A \left[\sum_{j=1}^q \left[\sum_{k=1}^{n_j} \alpha_k^j v_k^j \right] + \sum_{j=q+1}^p \alpha_1^j v_1^j \right] = 0$ and then

$$\sum_{j=1}^q \left[\alpha_k^j \lambda_j v_1^j + \sum_{k=2}^n \alpha_k^j (\lambda_j v_k^j + v_{k-1}^j) \right] + \alpha_1^j \lambda_j v_1^j = 0,$$

from where

$$(11) \quad \sum_{j=1}^{q-1} \lambda_j \left[\sum_{k=1}^{n_j} \alpha_k^j v_k^j \right] + \sum_{j=1}^{q-1} \left[\sum_{k=1}^{n_{j-1}} \alpha_{k+1}^j v_k^j \right] + \lambda_q \sum_{k=1}^{n_q} \alpha_k^q v_k^q + \\ + \sum_{k=1}^{n_{q-1}} \alpha_{k+1}^q v_k^q + \sum_{j=q+1}^p \alpha_j^j \lambda_j v_1^j = 0.$$

Replacing in (11) $\sum_{k=1}^{n_q} \alpha_k^q v_k^q$ from (10), we obtain

$$(12) \quad \sum_{j=1}^{q-1} (\lambda_j - \lambda_q) \sum_{k=1}^{n_j} \alpha_k^j v_k^j + \sum_{j=1}^q \left[\sum_{k=1}^{n_{j-1}} \alpha_{k+1}^j v_k^j \right] + \\ + \sum_{k=1}^{n_{q-1}} \alpha_{k+1}^q v_k^q + \sum_{j=q+1}^p (\lambda_j - \lambda_q) \alpha_1^j v_1^j = 0.$$

From (12), because of the induction hypotheses with respect to q and n_q , it follows

$$(13) \quad \alpha_2^q = \dots = \alpha_{n_q}^q = 0.$$

Replacing (13) in (10) it results

$$(14) \quad \sum_{j=1}^{q-1} \left[\sum_{k=1}^{n_j} \alpha_k^j v_k^j \right] + \alpha_1^q v_1^q + \sum_{j=q+1}^p \alpha_1^j v_1^j = 0.$$

From (14), taking into account the induction hypothesis with respect to q , we obtain also $\alpha_k^j = 0$, $k = \overline{1, n_j}$, $j = \overline{1, q-1}$, $\alpha_1^j = 0$, $j = \overline{q, p}$.

The following proposition results immediately.

Proposition 1. Consider the matrix $A = (a_{jk})_{n \times n}$, $a_{jk} \in \mathbb{R}$, and let λ_0 be a real root of the characteristic polynomial (1). If $\{v_1, \dots, v_p\}$ is a series

of eigen-vectors and associated vectors, corresponding to λ_0 , then the systems of vectors $\{\operatorname{Re} v_1, \dots, \operatorname{Re} v_p\}$, $\{\operatorname{Im} v_1, \dots, \operatorname{Im} v_p\}$ are also series of eigen-vectors and associated vectors, of the matrix A , corresponding to λ_0 .

L e m m a 1. Let consider

$$V = \{v_1, \dots, v_p\} \subset \mathbb{C}, \quad v_k = \begin{bmatrix} v_{1k} \\ \vdots \\ v_{nk} \end{bmatrix}, \quad k = \overline{1, p}$$

and

$$V' = \{v_1, \dots, v_{j-1}, \operatorname{Re} v_j, v_{j+1}, \dots, v_p\}, \quad V'' = \{v_1, \dots, v_{j-1}, \operatorname{Im} v_j, v_{j+1}, \dots, v_p\}.$$

If V is linearly independent then V' either V'' are linearly independent.

P r o o f. Consider the matrix $[v_1, \dots, v_p] = \begin{bmatrix} v_{11} & \dots & v_{1p} \\ \vdots & & \vdots \\ v_{n1} & \dots & v_{np} \end{bmatrix}$. Because V is

linearly independent, we have $\operatorname{rank} [v_1, \dots, v_p] = p$. Then, there exists a p -order minor

$$\Delta_p = \begin{vmatrix} v_{m_1 1} & \dots & v_{m_1 j-1} & v_{m_1 j} & v_{m_1 j+1} & \dots & v_{m_1 p} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ v_{m_p 1} & \dots & v_{m_p j-1} & v_{m_p j} & v_{m_p j+1} & \dots & v_{m_p p} \end{vmatrix} \neq 0.$$

We denote

$$\Delta_p' = \begin{vmatrix} v_{m_1 1} & \dots & v_{m_1 j-1} & \operatorname{Re} v_{m_j} & v_{m_1 j+1} & \dots & v_{m_1 p} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ v_{m_p 1} & \dots & v_{m_p j-1} & \operatorname{Re} v_{m_p} & v_{m_p j+1} & \dots & v_{m_p p} \end{vmatrix},$$

$$\Delta_p'' = \begin{vmatrix} v_{m_1 1} & \dots & v_{m_1 j-1} & \operatorname{Im} v_{m_j} & v_{m_1 j+1} & \dots & v_{m_1 p} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ v_{m_p 1} & \dots & v_{m_p j-1} & \operatorname{Im} v_{m_p} & v_{m_p j+1} & \dots & v_{m_p p} \end{vmatrix}.$$

Then $\Delta_p = \Delta_p' + i\Delta_p''$. Because $\Delta_p \neq 0$, it results that or $\Delta_p' \neq 0$ or $\Delta_p'' \neq 0$ and

consequently, it follows the statement of the proposition.

Applying succesively Lemma 1 for p -times we obtain

L e m m a 2. *If the system of vectors $V=\{v_1, \dots, v_p\} \subset \mathbb{C}_n$ is linearly independent, then one can find a system of vectors $U=\{u_1, \dots, u_p\}$, linearly independent, where $u_j = \operatorname{Re} v_j$ or $u_j = \operatorname{Im} v_j$, $j=\overline{1, p}$.*

From the Lemma 2 and the Proposition 1 it follows

P r o p o s i t i o n 2. *Consider the matrix $A=(a_{jk})_{n \times n}$, $a_{jk} \in \mathbb{R}$, and let λ_0 be a real root of the characteristic polynomial (1). We assume:*

i) *the systems of vectors $\{v_1^j, \dots, v_{n_j}^j\}$, $j=\overline{1, p}$, are series of eigen-vectors and associated vectors, corresponding to λ_0 ;*

ii) *the vectors $v_1^1, \dots, v_1^p$ are linearly independent.*

Then we can find the system of vectors $\{u_1^j, \dots, u_{n_j}^j\}$, $j=\overline{1, p}$, so that:

a) *u_k^j , $j=\overline{1, p}$, $k=\overline{1, n}$ has all the components real;*

b) *$u_1^1, \dots, u_1^p$ are linearly independent;*

c) *$\{u_1^j, \dots, u_{n_j}^j\}$, $j=\overline{1, p}$ are series of eigen-vectors and associated vectors, corresponding to λ_0 .*

Using the operation of conjugation in $\mathbb{C}$, we obtain

P r o p o s i t i o n 3. *Consider the matrix $A=(a_{jk})_{n \times n}$, $a_{jk} \in \mathbb{R}$, and we assume that the characteristic polynomial (1) has the complex conjugate roots $\mu_0, \bar{\mu}_0 \in \mathbb{C}$. Then:*

a) *if $v_1, \dots, v_k$ are linearly independent eigen-vectors, corresponding to μ_0 , then $\bar{v}_1, \dots, \bar{v}_k$ are linearly independent eigen-vectors corresponding to $\bar{\mu}_0$.*

b) *if $\{u_1, \dots, u_p\}$ is a series of eigen-vectors and associated vectors, corresponding to μ_0 , then $\{\bar{u}_1, \dots, \bar{u}_p\}$ is a series of eigen-vectors and associated vectors, corresponding to $\bar{\mu}_0$.*

P r o p o s i t i o n 4. *Consider the matrix $A=(a_{jk})_{n \times n}$, $a_{jk} \in \mathbb{R}$, and we suppose that $\lambda_1, \dots, \lambda_{k_1} \in \mathbb{R}$, $\mu_1, \dots, \mu_{k_2}$, $\bar{\mu}_1, \dots, \bar{\mu}_{k_2} \in \mathbb{C}$ are the different roots of the characteristic polynomial (1). If*

$$(15) \quad \left\{ a_1^1, \dots, a_{m_1}^1, \dots, a_1^p, \dots, a_{m_p}^p, b_1^1, \dots, b_{n_1}^1, \dots, \right. \\ \left. \dots, b_1^r, \dots, b_{n_r}^r, c_1^1, \dots, c_{q_1}^1, \dots, c_1^s, \dots, c_{q_s}^s \right\}$$

is a Jordan basis in $\mathbb{C}_n$, of the matrix A , where $\{a_1^j, \dots, a_{n_j}^j\}$, $j=\overline{1, p}$ are the series of eigen-vectors and associated vectors, corresponding to the real roots

$\lambda_1, \dots, \lambda_{k_1}, \{b_1^j, \dots, b_{n_j}^j\}, j=\overline{1, r}$, are the series of eigen-vectors and associated vectors, corresponding to the complex roots $\mu_1, \dots, \mu_{k_2}, \{c_1^j, \dots, c_{q_j}^j\}, j=\overline{1, s}$, are the series of eigen-vectors and associated vectors, corresponding to the complex roots $\bar{\mu}_1, \dots, \bar{\mu}_{k_2}$, then $n_1 + \dots + n_r = q_1 + \dots + q_s$.

P r o o f. Assume that

$$(16) \quad n_1 + \dots + n_r > q_1 + \dots + q_s.$$

Then we consider the system of vectors

$$(17) \quad \{a_1^1, \dots, a_{m_1}^1, \dots, a_1^p, \dots, a_{m_p}^p, b_1^1, \dots, b_{n_1}^1, \dots, \\ \dots, b_1^r, \dots, b_{n_r}^r, \bar{b}_1^1, \dots, \bar{b}_{n_1}^1, \dots, \bar{b}_1^r, \dots, \bar{b}_{n_r}^r\};$$

(17) consists of series of eigen-vectors and associated vectors, corresponding to different roots $\lambda_1, \dots, \lambda_{k_1}, \mu_1, \dots, \mu_{k_2}, \bar{\mu}_1, \dots, \bar{\mu}_{k_2}$ (Proposition 3).

Taking into account the Proposition 3 and a well known theorem concerning the eigen-vectors corresponding to different eigenvalues, we obtain that the vectors $a_1^1, \dots, a_{m_1}^1, b_1^1, \dots, b_{n_1}^1, \bar{b}_1^1, \dots, \bar{b}_{n_1}^1$ are linearly independent. Relatively to (17), if (16) is true, it results

$$(18) \quad m_1 + \dots + m_p + n_1 + \dots + n_r + n_1 + \dots + n_r > \\ > m_1 + \dots + m_p + n_1 + \dots + n_r + q_1 + \dots + q_s = n.$$

The fact that (17) is linearly independent in C_n is incompatible with (18), so that (16) is not possible.

Analogously we obtain that it is not possible to have $q_1 + \dots + q_s > n_1 + \dots + n_r$.

Theorem 2. A Jordan basis of the form (2) exists in C_n for a matrix $A = (a_{jk})_{n \times n}$, $a_{jk} \in \mathbb{R}$.

P r o o f. We start of a Jordan basis of the form (15), which exists [7], [8], and we apply the procedure of the Proposition 2 to the vectors $a_1^1, \dots, a_{m_1}^1, \dots, a_1^p, \dots, a_{m_p}^p$. In this way we obtain the system of vectors

$$(19) \quad \{u_1^1, \dots, u_{m_1}^1, \dots, u_1^p, \dots, u_{m_p}^p, b_1^1, \dots, b_{n_1}^1, \dots, \\ \dots, b_1^r, \dots, b_{n_r}^r, c_1^1, \dots, c_{q_1}^1, \dots, c_1^s, \dots, c_{q_s}^s\}.$$

From (19) we form the system of vectors (2) in the following way: the vectors $u_h^j, j=\overline{1, p}, h=\overline{1, n_j}$, are those of (19); the vectors $v_h^j, j=\overline{1, r}, h=\overline{1, n_j}$, are

$$(20) \quad v_h^j = b_h^j,$$

the vectors $w_h^j, j=\overline{1, r}, h=\overline{1, n_j}$, are

$$(21) \quad w_h^j = \bar{b}_h^j \quad \left(\text{the conjugate in } \mathbb{C} \text{ of } b_h^j \right).$$

The system of vectors (2), so determined, has the following properties: (2) consists of series of eigen-vectors and associated vectors (Propositions 2, 3); the vectors $u_1^1, \dots, u_1^p$ and $w_1^1, \dots, w_1^r$ are linearly independent (Propositions 2, 3); $m_1 + \dots + m_p + n_1 + \dots + n_r + n_1 + \dots + n_r = m_1 + \dots + m_r + n_1 + \dots + n_r + q_1 + \dots + q_s = n$. (Proposition 4).

Consequently, we can observe now that the system of vectors

$$(22) \quad \{u_1^1, \dots, u_1^p, v_1^1, \dots, v_1^r, w_1^1, \dots, w_1^r\}$$

can be separated in class of linearly independent eigen-vectors, corresponding to the different eigenvalues $\lambda_1, \dots, \lambda_{k_1}, \mu_1, \dots, \mu_{k_2}, \bar{\mu}_1, \dots, \bar{\mu}_{k_2}$, from where it result that (22) is linearly independent. As a consequence of Theorem 1, it follows that the system (2) so determined, is linearly independent, hence a basis in C_n .

3. A Fundamental System of Solutions with Real Components for a Linear Differential System with Constant Coefficients. We shall use a Jordan basis of the form (2) for to determine a fundamental system of solutions with real components, for the differential linear system with constant coefficients

$$(23) \quad x'(t) = Ax(t), \quad t \in \mathbb{R}, \quad A = (a_{jk})_{n \times n}, \quad a_{jk} \in \mathbb{R}, \quad x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}.$$

Concerning the system (23) we have:

(P₁) If $x(t) = \Re ex(t) + i \Im mx(t)$, $t \in \mathbb{R}$ is a solution of the system (23), then $\Re ex(t)$ and $\Im mx(t)$, $t \in \mathbb{R}$ are solutions of the system (23).

(P₂) If $\{v_1, \dots, v_k\} \subset C_n$ is a series of eigen-vectors and associated vectors of the matrix A , corresponding to eigenvalue λ_0 , then

$$(24) \quad \begin{aligned} x_1(t) &= e^{\lambda_0 t} v_1, \quad x_2(t) = e^{\lambda_0 t} \left[\frac{t}{1!} v_1 + v_2 \right], \dots, \\ x_k(t) &= e^{\lambda_0 t} \left[\frac{t^{k-1}}{(k-1)!} v_1 + \frac{t^{k-2}}{(k-2)!} v_2 + \dots + \frac{t}{1!} v_{k-1} + v_k \right], \quad t \in \mathbb{R}, \end{aligned}$$

are solution of the system (23).

We consider a Jordan basis in C_n , for the matrix A , of the form (2). Then, as a consequence of (P₂), the functions

$$\begin{aligned}
 x_1^j(t) &= e^{\lambda_j t} u_1^j, \quad x_h^j(t) = e^{\lambda_j t} \left[\frac{t^{h-1}}{(h-1)!} u_1^j + \frac{t^{h-2}}{(h-2)!} u_2^j + \dots + \frac{t}{1!} u_{h-1}^j + u_h^j \right], \\
 j &= \overline{1, p}, \quad h = \overline{2, n_j}; \quad y_1^j(t) = e^{\mu_j t} v_1^j, \\
 (25) \quad y_h^j(t) &= e^{\mu_j t} \left[\frac{t^{h-1}}{(h-1)!} v_1^j + \frac{t^{h-2}}{(h-2)!} v_2^j + \dots + \frac{t}{1!} v_{h-1}^j + v_h^j \right], \quad j = \overline{1, r}, \quad h = \overline{2, n_j}; \\
 z_1^j(t) &= e^{\bar{\mu}_j t} w_1^j, \quad z_h^j(t) = e^{\bar{\mu}_j t} \left[\frac{t^{h-1}}{(h-1)!} w_1^j + \frac{t^{h-2}}{(h-2)!} w_2^j + \dots + \frac{t}{1!} w_{h-1}^j + w_h^j \right], \\
 j &= \overline{1, r}, \quad h = \overline{2, n_j}.
 \end{aligned}$$

Obviously, $z_h^j(t) = \bar{y}_h^j(t)$, $j = \overline{1, r}$, $h = \overline{1, n_j}$.

We denote $W(t)$ the Wronskian of the system

$$\begin{aligned}
 (25') \quad & \{x_1^1, \dots, x_{m_1}^1, \dots, x_1^p, \dots, x_{m_p}^p, y_1^1, \dots, y_{n_1}^1, \dots, \\
 & \dots, y_1^r, \dots, y_{n_r}^r, z_1^1, \dots, z_{n_1}^1, \dots, z_1^r, \dots, z_{n_r}^r\}.
 \end{aligned}$$

Then, because (2) is a basis in $\mathbb{C}_n$, we have

$$\begin{aligned}
 W(0) &= \left| \begin{bmatrix} u_1^1, \dots, u_{m_1}^1, \dots, u_1^p, \dots, u_{m_p}^p, v_1^1, \dots, v_{n_1}^1, \dots, \\ \dots, v_1^r, \dots, v_{n_r}^r, w_1^1, \dots, w_{n_1}^1, \dots, w_1^r, \dots, w_{n_r}^r \end{bmatrix} \right| \neq 0,
 \end{aligned}$$

hence (25') is a fundamental system of solutions of (23) [8], [9]. Having (25') we can consider the system of solutions

$$(26) \quad x_1^1, \dots, x_{n_p}^p, \operatorname{Re} y_1^1, y_2^1, \dots, y_{n_1}^1, \dots, y_1^r, \dots, y_{m_r}^r,$$

$$\operatorname{Im} y_1^1, z_2^1, \dots, z_{m_1}^1, \dots, z_1^r, \dots, z_{m_r}^r.$$

If we denote $W_1(t)$ the Wronskian of the system (26), it results immediately that $W(0) = -2iW_1(0)$, hence $W_1(0) \neq 0$ and consequently, (26) is a new fundamental system of solutions of (23).

In a same way we obtain that

$$(27) \quad \left\{ x_1^1, \dots, x_{n_p}^p, \operatorname{Re} y_1^1, \operatorname{Re} y_2^1, y_3^1, \dots, y_{n_1}^1, \dots, y_1^r, \dots, y_{m_r}^r, \right. \\ \left. \operatorname{Im} y_1^1, \operatorname{Im} y_2^1, z_3^1, \dots, z_{m_1}^1, \dots, z_1^r, \dots, z_{n_r}^r \right\}$$

is a fundamental system of solutions of (23) and so on.

Thus we proved the

Theorem 3. *The system of solutions*

$$(28) \quad x_1^1, \dots, x_{m_1}^1, \dots, x_1^p, \dots, x_{m_p}^p, \operatorname{Re} y_1^1, \dots, \operatorname{Re} y_{n_1}^1, \dots, \operatorname{Re} y_1^r, \dots, \\ \operatorname{Re} y_{n_r}^r, \operatorname{Im} y_1^1, \dots, \operatorname{Im} y_{n_1}^1, \dots, \operatorname{Im} y_1^r, \dots, \operatorname{Im} y_{n_r}^r$$

is a fundamental system of solutions of the system (23).

Received March 31, 1993

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BAZE JORDAN ȘI SISTEME DE ECUAȚII DIFERENȚIALE LINIARE CU COEFICIENȚI CONSTANȚI

(Rezumat)

Pentru o matrice $A = (a_{jk})_{n \times n}$, $a_{jk} \in \mathbb{R}$, se demonstrează, elementar, existența unor baze Jordan în $\mathbb{C}_n$ de forma (2); o astfel de bază se utilizează, apoi, la determinarea unui sistem fundamental de soluții pentru sistemul de ecuații diferențiale liniare cu coeficienți constanți $x' = Ax$.

NONLINEAR DISCRETE GENERALIZATIONS OF GRONWALL'S INEQUALITY

BY

SEVER SILVESTRU DRAGOMIR

In paper [5] (see also [1] Lemma 1) B. G. Pachpatte proved the following discrete lemma of Gronwall type:

L e m m a 1. *Let $x(n)$, $f(n)$, $g(n)$ and $h(n)$ be real valued functions defined for $n \in \mathbb{N}$ with $g(n)$ and $h(n)$ nonnegative. If*

$$(1) \quad x(n) \leq f(n) + g(n) \sum_{s=0}^{n-1} h(s)x(s), \quad n \geq 1,$$

then

$$(2) \quad x(n) \leq f(n) + g(n) \sum_{s=0}^{n-1} h(s)f(s) \prod_{\tau=s+1}^{n-1} (h(\tau)g(\tau) + 1)$$

for all $n \in \mathbb{N}$.

Further, we shall give the discrete analogue of the nonlinear integral inequality

$$(3) \quad x(t) \leq A(t) + B(t) \int_{\alpha}^t L(s, x(s)) ds, \quad t \in [\alpha, \beta) \subset \mathbb{R},$$

where all the involved functions are continuous and nonnegative where they are defined and the mapping L satisfies the condition

$$(L) \quad 0 \leq L(t, u) - L(t, v) \leq M(t, v)(u - v), \quad t \in [\alpha, \beta), \quad u \geq v \geq 0$$

and M is nonnegative continuous on $[\alpha, \beta) \times \mathbb{R}_+$.

For the estimations of the nonnegative continuous solutions of (3) we send to [2], [3] and [4] where further consequences and applications in integral and differential equations theory are given.

Theorem 1. Let $A(n)$, $B(n)$ be nonnegative sequences and $L: \mathbb{N} \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with the property:

$$(L, n) \quad 0 \leq L(n, x) - L(n, y) \leq M(n, y)(x - y) \text{ for } n \in \mathbb{N}, \\ x \geq y \geq 0 \text{ and } M \text{ is nonnegative on } \mathbb{N} \times \mathbb{R}_+.$$

If $x(n) \geq 0$ ($n \in \mathbb{N}$) and

$$(4) \quad x(n) \leq A(n) + B(n) \sum_{s=0}^{n-1} L(s, x(s)) \text{ for } n \geq 1,$$

then the following estimation

$$(5) \quad x(n) \leq A(n) + B(n) \sum_{s=0}^{n-1} L(s, A(s)) \prod_{\tau=s+1}^{n-1} (M(\tau, A(\tau))B(\tau) + 1)$$

for $n \geq 1$, holds.

Proof. Put $y(n) := \sum_{s=0}^{n-1} L(s, x(s))$ for $n \geq 1$ and $y(0) = 0$.

Then we have

$$y(n+1) - y(n) = L(n, x(n)) \leq L(n, A(n) + B(n)y(n)) \leq \\ \leq L(n, A(n)) + M(n, A(n))B(n)y(n)$$

for $n \geq 0$, i.e.,

$$y(n+1) \leq L(n, A(n)) + [M(n, A(n))B(n) + 1]y(n), \quad n \geq 0.$$

Using the notations

$$\alpha(n) := L(n, A(n)), \quad \beta(n) := M(n, A(n))B(n) + 1 > 0, \quad n \geq 0,$$

$$z(n) := \frac{y(n)}{\prod_{\tau=0}^{n-1} \beta(\tau)}, \quad n \geq 1, \quad z(0) = 0,$$

we can write

$$\left[\prod_{\tau=0}^n \beta(\tau) \right] z(n+1) \leq \alpha(n) + \left[\prod_{\tau=0}^n \beta(\tau) \right] z(n),$$

and since $\prod_{\tau=0}^n \beta(\tau) > 0$ ($n \geq 0$), we obtain

$$z(n+1) - z(n) \leq \frac{\alpha(n)}{\prod_{\tau=0}^n \beta(\tau)}, \quad n \geq 0.$$

Adding these inequalities, we deduce

$$z(n) \leq \frac{\sum_{s=0}^{n-1} \alpha(s)}{\prod_{\tau=0}^s \beta(\tau)},$$

what implies

$$y(n) \leq \sum_{s=0}^{n-1} L(s, A(s)) \prod_{\tau=s+1}^{n-1} (M(\tau, A(\tau))B(\tau) + 1)$$

for all $n \geq 1$.

The theorem is proved.

COROLLARY 1.1. Let $A(n)$, $B(n)$ be nonnegative sequences and $G: \mathbb{N} \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ a mapping satisfying the condition:

$$(G, n) \quad 0 \leq G(n, x) - G(n, y) \leq N(n)(x - y), \quad n \in \mathbb{N}, \quad x \geq y \geq 0$$

and $N(n)$ is nonnegative for $n \in \mathbb{N}$.

If $x(n) \geq 0$ and

$$(6) \quad x(n) \leq A(n) + B(n) \sum_{s=0}^{n-1} G(s, x(s)), \quad \text{for } n \geq 1,$$

then the following evaluation

$$(7) \quad x(n) \leq A(n) + B(n) \sum_{s=0}^{n-1} G(s, A(s)) \prod_{\tau=s+1}^{n-1} (N(\tau)B(\tau) + 1), \quad n \geq 1,$$

holds.

The statement follows directly from the above theorem.

COROLLARY 1.2. Let $A(n)$, $B(n)$, $C(n)$ be nonnegative sequences and $H: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ a function satisfying the following Lipschitz type condition:

$$(H) \quad 0 \leq H(x) - H(y) \leq M(x - y),$$

where $M > 0$ and $u \geq v \geq 0$.

Then for any $x(n)$ nonnegative sequence verifying the condition

$$(8) \quad x(n) \leq A(n) + B(n) \sum_{s=0}^{n-1} C(s)H(x(s)), \quad n \geq 1,$$

we have

$$(9) \quad x(n) \leq A(n) + B(n) \sum_{s=0}^{n-1} C(s)H(A(s)) \prod_{\tau=s+1}^{n-1} (MC(\tau)B(\tau) + 1)$$

for all $n \geq 1$.

The proof is obviously and we omit the details.

Remark. If in the previous corollary we put

$$A(n) = f(n), B(n) = g(n), C(n) = h(n) \quad (n \in \mathbb{N}) \text{ and } H(x) = x,$$

we obtain the lemma of B. G. Pachpatte.

Now, we shall point out the second main result of our paper.

Theorem 2. Let $A(n), B(n)$ be nonnegative sequences and $D: \mathbb{N} \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ a mapping satisfying the condition:

for any $n \in \mathbb{N}$. $D(n, \cdot)$ is differentiable on $\mathbb{R}_+$, $dD(n, t)/dt$ is non-negative for $n \in \mathbb{N}, t \in \mathbb{R}_+$ and there exists a function $P: \mathbb{N} \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $dD(n, u)/dt \leq P(n, v)$ for $n \in \mathbb{N}$, and $u \geq v \geq 0$.

If $x(n)$ is nonnegative and verifies the inequality

$$(10) \quad x(n) \leq A(n) + B(n) \sum_{s=0}^{n-1} D(s, x(s)), \quad n \geq 1,$$

then the following estimation is valid

$$(11) \quad x(n) \leq A(n) + B(n) \sum_{s=0}^{n-1} D(s, A(s)) \prod_{\tau=s+1}^{n-1} (P(\tau, A(\tau))B(\tau) + 1)$$

for all $n \geq 1$.

Proof. Let $n \in \mathbb{N}$. Then by Lagrange's theorem, for any $u > v \geq 0$, then there exists $\mu_n \in (v, u)$ such that

$$D(n, u) - D(n, v) = \frac{dD(n, \mu_n)}{dt} (u - v).$$

Since $0 \leq dD(n, \mu_n)/dt \leq P(n, v)$, we obtain

$$0 \leq D(n, u) - D(n, v) \leq P(n, v)(u - v) \text{ for } u \geq v \geq 0 \text{ and } n \in \mathbb{N}.$$

Now, the proof of theorem follows by an argument similar to that in the proof of Theorem 1, and we omit the details.

Corollary 2.1. Let $A(n), B(n)$ be nonnegative for $n \geq 1$ and $I: \mathbb{N} \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that:

(I, n) $I(n, \cdot)$ is differentiable on $\mathbb{R}_+$ for $n \in \mathbb{N}$, $dI(n, t)/dt$ is nonnegative on $\mathbb{N} \times \mathbb{R}_+$ and $dI(n, u)/dt \leq dI(n, v)/dt$ for $u \geq v \geq 0$ and $n \in \mathbb{N}$.

Then for any $x(n)$ satisfying the inequality

$$(12) \quad x(n) \leq A(n) + B(n) \sum_{s=0}^{n-1} I(s, x(s)), \quad n \geq 1,$$

we have

$$(13) \quad x(n) \leq A(n) + B(n) \sum_{s=0}^{n-1} I(s, A(s)) \prod_{\tau=s+1}^{n-1} \left[\frac{dI(\tau, A(\tau))}{dt} B(\tau) + 1 \right]$$

for all $n \geq 1$.

The proof is obvious. We omit the details.

Finally, we have

Corollary 2.2. Let $A(n)$, $B(n)$, $C(n)$ be nonnegative sequence and $K: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ a mapping with the property:

(K) K is nondecreasing and differentiable on $\mathbb{R}_+$ with $dK/dr: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is nonincreasing on $\mathbb{R}_+$.

If $x(n)$ is nonnegative and

$$(14) \quad x(n) \leq A(n) + B(n) \sum_{s=0}^{n-1} C(s)K(x(s)), \quad n \geq 1,$$

then

$$(15) \quad x(n) \leq A(n) + B(n) \sum_{s=0}^{n-1} C(s)K(A(s)) \prod_{\tau=s+1}^{n-1} \left[\frac{dK}{dr}(A(\tau))B(\tau) + 1 \right]$$

for all $n \geq 1$.

Now, we can give some natural consequences of the above Corollaries.

Consequences 1. Let $A(n)$, $B(n)$, $C(n)$, $r(n)$ and $x(n)$ be nonnegative sequences and $A(n) > 0$, $r(n) \leq 1$ for $n \geq 1$. If

$$(r, n) \quad x(n) \leq A(n) + B(n) \sum_{s=0}^{n-1} C(s)x(s)^{r(s)}, \quad n \geq 1,$$

then

$$(16) \quad x(n) \leq A(n) + B(n) \sum_{s=0}^{n-1} C(s)A(s)^{r(s)} \prod_{\tau=s+1}^{n-1} \left[\frac{rC(\tau)B(\tau)}{A(\tau)^{1-r(\tau)}} + 1 \right]$$

for all $n \geq 1$.

Particularly, if $r \in [0, 1)$, then

$$(rc, n) \quad x(n) \leq A(n) + B(n) \sum_{s=0}^{n-1} C(s)x(s)^r, \quad n \geq 1,$$

implies

$$(17) \quad x(n) \leq A(n) + B(n) \sum_{s=0}^{n-1} C(s)A(s)^r \prod_{\tau=s+1}^{n-1} \left[\frac{rC(\tau)B(\tau)}{A(\tau)^{1-r}} + 1 \right]$$

for all $n \geq 1$.

2. If $A(n)$, $B(n)$, $C(n)$ and $x(n)$ are nonnegative sequences and

$$(ln, n) \quad x(n) \leq A(n) + B(n) \sum_{s=0}^{n-1} C(s) \ln(x(s) + 1),$$

then we have the estimation

$$(18) \quad x(n) \leq A(n) + B(n) \sum_{s=0}^{n-1} C(s) \ln(A(s)+1) \prod_{\tau=s+1}^{n-1} \left[\frac{C(\tau)B(\tau)}{A(\tau)+1} + 1 \right]$$

for all $n \geq 1$.

3. Assume that $A(n)$, $B(n)$, $C(n)$ and $x(n)$ are as above and

$$(\tan^{-1}, n) \quad x(n) \leq A(n) + B(n) \sum_{s=0}^{n-1} C(s) \tan^{-1}(x(s)), \quad n \geq 1,$$

then the following inequality is true

$$(19) \quad x(n) \leq A(n) + B(n) \sum_{s=0}^{n-1} C(s) \tan^{-1}(A(s)) \prod_{\tau=s+1}^{n-1} \left[\frac{B(\tau)C(\tau)}{A^2(\tau)+1} + 1 \right]$$

for all $n \geq 1$.

Acknowledgement. The author is indebted to Professor Adrian C o r d u n e a n u of Polytechnic Institute of Jassy for his help in the preparation of this paper.

Received November 14, 1990

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GENERALIZĂRI NELINIARE DISCRETE ALE INEGALITĂȚII GRONWALL

(Rezumat)

Se stabilesc analogele discrete ale unor inegalități integrale de tip Gronwall stabilite în [2], [3] și [4] și se dau câteva cazuri particulare care permit regăsirea unei binecunoscute inegalități a lui B. G. P a c h p a t t e [5].

PARTIALLY DERIVATION OPERATORS OF SECOND ORDER ON A CLASS OF MULTIVALUED FUNCTIONS

BY

M. GONTINEAC

1. Introduction. [1] dealt with a class of multivalued functions defined on the cartesian product $Q_+^2 = Q_+ \times Q_+$, where Q_+ is the set of positive rational numbers and valued in the set of real sequences.

On this class of functions we have defined a class of linear operators, called partially derivation operators, and we have proved some properties of them.

By such a multivalued function, to each ordered pair (mn, nh) of positive rational numbers, corresponds a real sequence denoted $a_{mh, nh}^k$, where k are the indices of terms of the sequence, $(m, n) \in \mathbb{N}^2$, $k \in \mathbb{N}^*$ and $h = (k!)^{-1}$ is the step of the sequence.

Definition 1. An operator denoted by $\nabla_p^m a_{mh, nh}^k$, is called a p -mode partially derivation operator of first order, of the sequence $a_{mh, nh}^k$, with respect to the index mh , if

$$(1) \quad \nabla_p^m a_{mh, nh}^k = \sum_{i=0}^p \alpha_i a_{(m+1)h, nh}^k,$$

where $\alpha_0, \alpha_1, \dots, \alpha_p$ are real numbers satisfying

$$(1.1) \quad \sum_{i=1}^p \alpha_i = 0,$$

$$(1.2) \quad \sum_{i=1}^p i \alpha_i = \frac{1}{h}, \quad h = \frac{1}{k!}.$$

Definition 2. An operator denoted by $\nabla_q^n a_{mh, nh}^k$ is called a q -mode partially operator of first order, of the sequence $a_{mh, nh}^k$ with respect to index nh , if

$$(2) \quad \nabla_q^n a_{mh, nh}^k = \sum_{i=0}^q \alpha_i a_{mh, (n+i)h}^k,$$

where $\alpha_0, \alpha_1, \dots, \alpha_p$ are real numbers satisfying conditions of the same type as (1.1) and (1.2).

Remark. The sum and the product with real scalars of partially derivation operators are defined in an usual manner. Let us remark that a linear combination of partially derivation operators may be not a partially derivation operator.

2. Partially Derivation Operators of Second Order. Definition

3. An operator denoted by $\nabla_p^2 a_{mh, nh}^k$ is called a p -mode partially derivation operator of second order, of the sequence $a_{mh, nh}^k$, with respect to the index mh , if

$$(3) \quad \nabla_p^2 a_{mh, nh}^k = \sum_{i=0}^p \alpha_i a_{(m+i)h, nh}^k,$$

where $\alpha_0, \alpha_1, \dots, \alpha_p$ are real numbers satisfying

$$(3.1) \quad \sum_{i=0}^p \alpha_i = 0,$$

$$(3.2) \quad \sum_{i=0}^p i \alpha_i = 0,$$

$$(3.3) \quad \sum_{i=0}^p i^2 \alpha_i = \frac{2}{h^2}.$$

Definition 4. An operator denoted by $\nabla_q^2 a_{mh, nh}^k$ is called a q -mode partially derivation operator of second order, of the sequence $a_{mh, nh}^k$, with respect to the index nh , if

$$(4) \quad \nabla_q^2 a_{mh, nh}^k = \sum_{i=0}^q \alpha_i a_{mh, (n+i)h}^k,$$

where $\alpha_0, \alpha_1, \dots, \alpha_p$ are real numbers satisfying relations similar to (3.1), (3.2) and (3.3).

Definition 5. An operator denoted by $\delta_{p_m}^2$ is called a p -mode basic partially derivation operator of second order, of the sequence $a_{mh, nh}^k$,

with respect to the index nh , if

$$(5) \quad \delta_{p_m}^2 a_{mh, nh}^k = \frac{a_{(m+p+1)h, nh}^k - a_{(m+p)h, nh}^k - a_{(m+1)h, nh}^k + a_{mh, nh}^k}{ph^2}.$$

Definition 6. An operator denoted by $\delta_{q_n}^2$ is called a q -mode basic partially derivation operator of second order, of the sequence $a_{mh, nh}^k$, with respect to the index nh , if

$$(6) \quad \delta_{q_n}^2 a_{mh, nh}^k = \frac{a_{mh, (n+q+1)h}^k - a_{mh, (n+q)h}^k - a_{mh, (n+1)h}^k + a_{mh, nh}^k}{qh^2}.$$

Lemma 1. A linear combination of two partially derivation operators of second order of $a_{mh, nh}^k$, whose sum of coefficients equals 1, is a partially derivation operator of second order of $a_{mh, nh}^k$.

Proof. Let $\nabla_{p_1}^2 a_{mh, nh}^k$, $j=1,2$, two partially derivation operators of second order of $a_{mh, nh}^k$. We shall prove that

$$\lambda \nabla_{p_1}^2 a_{mh, nh}^k + (1-\lambda) \nabla_{p_2}^2 a_{mh, nh}^k$$

is a partially derivation operator of second order of $a_{mh, nh}^k$.

In view of the Definition 3, we have

$$\nabla_{p_j}^2 a_{mh, nh}^k = \sum_{i=0}^{p_j} \alpha_{ij} a_{(m+i)h, nh}^k, \quad j=1,2,$$

where

$$(3.4) \quad \sum_{i=0}^{p_j} \alpha_{ij} = 0, \quad j=1,2,$$

$$(3.5) \quad \sum_{i=0}^{p_j} i \alpha_{ij} = 0, \quad j=1,2,$$

$$(3.6) \quad \sum_{i=0}^{p_j} i^2 \alpha_{ij} = \frac{2}{h^2}, \quad j=1,2.$$

Then

$$\begin{aligned}
 (7) \quad & \lambda \nabla_{p_1}^2 a_{mh, nh}^k + (1 - \lambda) \nabla_{p_2}^2 a_{mh, nh}^k = \\
 & = \lambda \sum_{i=0}^{p_1} \alpha_{i1} a_{(m+i)h, nh}^k + (1 - \lambda) \sum_{i=0}^{p_2} \alpha_{i2} a_{(m+i)h, nh}^k,
 \end{aligned}$$

where

$$\lambda \sum_{i=0}^{p_1} \alpha_{i1} + (1 - \lambda) \sum_{i=0}^{p_2} \alpha_{i2} = 0,$$

$$\lambda \sum_{i=0}^{p_1} i \alpha_{i1} + (1 - \lambda) \sum_{i=0}^{p_2} i \alpha_{i2} = 0$$

and

$$\lambda \sum_{i=0}^{p_1} i^2 \alpha_{i1} + (1 - \lambda) \sum_{i=0}^{p_2} i^2 \alpha_{i2} = \frac{2}{h^2}$$

as consequence of the relations (3.4), (3.5) and (3.6). Therefore, the coefficients of $a_{mh, nh}^k$ in (7), satisfying conditions similar to (3.1), (3.2) and (3.3), the operator defined by (7) is a partially derivation operator of second order of $a_{mh, nh}^k$. Q. E. D.

Proposition 1. A linear combination of s , $s \in \mathbb{N}$, partially derivation operators of second order of $a_{mh, nh}^k$, whose sum of coefficients equals 1, is a partially derivation operator of second order of $a_{mh, nh}^k$.

P r o o f. Let $\nabla_{p_j}^2 a_{mh, nh}^k$, $j \in \{1, 2, \dots, s\}$, be a partially derivation operator of second order of $a_{mh, nh}^k$. Using the induction method, we shall prove that

$$\sum_{j=1}^s \lambda_j \nabla_{p_j}^2 a_{mh, nh}^k \quad \text{satisfying} \quad \sum_{j=1}^s \lambda_j = 1$$

is a partially derivation operator of second order of $a_{mh, nh}^k$.

In view of Lemma 1, the conclusion is true for $s=2$. Suppose that

$$\sum_{j=1}^{s-1} \lambda_j' \nabla_{p_j}^2 a_{mh, nh}^k \quad \text{satisfying} \quad \sum_{j=1}^{s-1} \lambda_j' = 1$$

is a partially derivation operator of second order of $a_{mh, nh}^k$. Then

$$\sum_{j=1}^s \lambda_j \nabla_{p_j}^2 a_{mh, nh}^k = \sum_{j=1}^{s-1} \lambda_j \nabla_{p_j}^2 a_{mh, nh}^k + \lambda_s \nabla_{p_s}^2 a_{mh, nh}^k =$$

$$= \mu \sum_{j=1}^{s-1} \lambda_j' \nabla_{p_j}^2 a_{mh, nh}^k + \lambda_s \nabla_{p_s}^2 a_{mh, nh}^k,$$

where, without loss of generality, for $\sum_{j=1}^{s-1} \lambda_j \neq 0$,

$$(8) \quad \lambda_j' = \frac{\lambda_j}{\mu}, \quad \mu = \sum_{j=1}^{s-1} \lambda_j.$$

From (8), it follows $\sum_{j=1}^{s-1} \lambda_j' = 1$, and, in view of the inductive hypothesis

$$\sum_{j=1}^{s-1} \lambda_j' \nabla_{p_j}^2 a_{mh, nh}^k$$

is a partially derivation operator of second order of $a_{mh, nh}^k$. Moreover, by hypothesis, $\mu + \lambda_s = 1$ and, in view of Lemma 1,

$$\sum_{j=1}^s \lambda_j \nabla_{p_j}^2 a_{mh, nh}^k$$

is a partially derivation operator of second order of $a_{mh, nh}^k$. Q. E. D.

Remark. Lemma 1 and Proposition 1 are also true for the operators defined by (4).

Proposition 2. *The composite function of two partially derivation operators of first order of $a_{mh, nh}^k$ is a partially derivation operator of second order of $a_{mh, nh}^k$.*

P r o o f. To each term of the sequence $a_{mh, nh}^k$ apply the first order operator ∇_p^2 and obtain

$$\nabla_p^2 a_{mh, nh}^k = \sum_{i=0}^p \alpha_i a_{(m+i)h, nh}^k,$$

the coefficients α_i satisfying (1.1) and (1.2). To each term of the sequence of operators $\nabla_p^2 a_{mh, nh}^k$ apply the first order operator ∇_s and obtain

$$\begin{aligned}
 (9) \quad \nabla_s^m \left(\nabla_p^m a_{mh, nh}^k \right) &= \sum_{j=0}^s \beta_j \nabla_p^m a_{(m+j)h, nh}^k = \sum_{j=0}^s \beta_j \sum_{i=1}^p \alpha_i a_{(m+i+j)h, nh}^k = \\
 &= \sum_{j=0}^s \sum_{i=1}^p \beta_j \alpha_i a_{(m+i+j)h, nh}^k,
 \end{aligned}$$

the coefficients satisfying the conditions

$$(9.1) \quad \sum_{j=0}^s \beta_j = 0,$$

$$(9.2) \quad \sum_{j=0}^s j \beta_j = 1.$$

To prove that $\nabla_s^m \left(\nabla_p^m a_{mh, nh}^k \right)$ is a partially derivation operator of second order of $a_{mh, nh}^k$, we shall show that the coefficients α_i, β_j satisfy the conditions (3.1), (3.2) and (3.3).

From (1.1) or (9.1) we have

$$\sum_{j=0}^s \sum_{i=0}^p \beta_j \alpha_i = \sum_{i=0}^s \beta_j \sum_{i=0}^p \alpha_i = 0;$$

also, from (1.1), (1.2), (9.1) and (9.2) we have

$$\sum_{j=0}^s \sum_{i=0}^p (i+j) \beta_j \alpha_i = \sum_{j=0}^s \beta_j \sum_{i=0}^p i \alpha_i + \sum_{j=0}^s j \beta_j \sum_{i=0}^p \alpha_i = 0;$$

finally, from (1.1), (1.2), (9.1) and (9.2) we have

$$\sum_{j=0}^s \sum_{i=0}^p (i+j)^2 \beta_j \alpha_i = \sum_{j=0}^s \beta_j \sum_{i=0}^p i^2 \alpha_i + 2 \sum_{j=0}^s j \beta_j \sum_{i=0}^p i \alpha_i + \sum_{j=0}^s j^2 \beta_j \sum_{i=0}^p \alpha_i = 0.$$

Remark 1. The relation (9) implies the permutability of the composition of operators of first order.

Remark 2. Proposition 1 is true for the operators given by (4).

Proposition 3. Every partially derivation operator of second order of $a_{mh, nh}^k$ is a unique linear combination, the sum of whose coefficients equals 1, of basic partially derivation operator of second order of $a_{mh, nh}^k$.

Proof. Let

$$\nabla_p^m a_{mh, nh}^k = \sum_{i=0}^p \alpha_i a_{(m+i)h, nh}^k$$

be, the coefficients α_i satisfying (3.1), (3.2) and (3.3). From (5) we deduce

$$(10) \quad a_{(m+p+1)h, nh}^k = p h^2 \delta_{pm}^2 a_{mh, nh}^k + a_{(m+p)h, nh}^k + a_{(m+1)h, nh}^k - a_{mh, nh}^k.$$

In (10) put $p = 1, p = 2, \dots, 2$ instead of p and add

$$a_{(m+i)h, nh}^k = \sum_{j=1}^{i-1} h^2 (i-j) \delta_{im}^2 - j a_{mh, nh}^k + i a_{(m+1)h, nh}^k - (i-1) a_{mh, nh}^k.$$

Then

$$\begin{aligned} \nabla_p^2 a_{mh, nh}^k &= \sum_{i=1}^{p-1} \sum_{j=i+1}^p h^2 i \alpha_j \delta_{im}^2 a_{mh, nh}^k - \\ &\quad - a_{mh, nh}^k \sum_{i=0}^p (i-1) \alpha_i + a_{(m+1)h, nh}^k \sum_{i=1}^p i \alpha_i. \end{aligned}$$

From (3.1) and (3.2) obtain $\sum_{i=0}^p (i-1) \alpha_i = 0$ and $\sum_{i=0}^p i \alpha_i = 0$ and, therefore

$$(11) \quad \nabla_p^2 a_{mh, nh}^k = \sum_{i=1}^{p-1} \sum_{j=i+1}^p h^2 i \alpha_j \delta_{im}^2 a_{mh, nh}^k.$$

Let us show that the sum of coefficients of operators of second order $\delta_{im}^2 a_{mh, nh}^k$ is 1. We have

$$s = \sum_{i=1}^{p-1} \sum_{j=i+1}^p h^2 i \alpha_j = \alpha_2 + (1+2)\alpha_3 + \dots + [1+2+\dots+(p-1)]\alpha_p = \frac{1}{2} \sum_{i=1}^p h^2 (i-1)i.$$

From (3.2) and (3.3) it follows $\sum_{i=1}^p i^2 \alpha_i - \sum_{i=1}^p i \alpha_i = \frac{2}{h^2}$ such that $s = \frac{h^2}{2} \frac{2}{h^2} = 1$.

The unicity of decomposition is obvious. Q. E. D.

Remark. The definitions and propositions given above can be extended to multivalued functions on $Q_+^n, n \in \mathbb{N}$.

Received January 26, 1993

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OPERATORI DE DERIVARE PARȚIALĂ DE ORDIN DOI PE O CLASĂ DE FUNCȚII MULTIVOCE

(Rezumat)

Se pune în evidență o clasă de operatori definiți pe o clasă de funcții multivoce și o serie de proprietăți ale acestor operatori.

A METHOD FOR APPROXIMATING THE SOLUTION TO A FINITE DIMENSIONAL VARIATIONAL INEQUALITY

BY

CĂTĂLINA DAVIDEANU

In this article we consider the finite dimensional variational inequality
 $(V_K) \quad (y''(t) + (F + \partial I_K)(y'(t)) \ni Bu(t) + f(t) \quad \text{a.e.} \quad t \in]0, T[,$

with the initial condition

(IC) $y''(0) = y_0 \in K.$

The elements which appear have the following properties:

(i) $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a locally Lipschitz function which satisfies the condition:
 there exist the constants $D_1, D_2 \in \mathbb{R}_+$ such that

$$(1) \quad (Fy, y) \geq -D_1 \|y\|^2 - D_2 \quad \text{for every } y \in \mathbb{R}^n;$$

(ii) ∂I_K is the subdifferential of the indicator function of the closed and convex set $K \subset \mathbb{R}^n$, with $0 \in K$;

(iii) $B: \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a linear continuous operator;

(iv) $f \in L^2(0, T; \mathbb{R}^m).$

For every $u \in L^2(0, T; \mathbb{R}^m)$, and $y \in K$ the finite dimensional variational inequality (V_K) with the initial condition (IC) admits a unique solution $y''(t) = y(t, y_0, u) \in W^{1,2}(0, T; \mathbb{R}^n)$ as it constitutes a particular case of the multivalued differential equation (E), which has been studied in [3].

The closed interval $[0, T]$ is divided into l intervals of length $d = T/l$ by the equidistant points

$$0 = t_0 \leq t_1 \leq \dots \leq t_{l-1} \leq t_l = T, \quad \text{where } t_i = id, \quad i = \overline{0, l}.$$

We construct, for every $d > 0$, the approximative differential equation

$$\left(E_d^i\right) \quad \left(y_d^u(t)\right)' + F\left(y_d^u(t)\right) = Bu(t) + f(t) \quad \text{a. e.} \quad t \in]id, (i+1)d[$$

with the initial conditions

$$(2) \quad \left(y_d^u\right)^+(id) = z_i(d)$$

for $i=1, \dots, l-1$, where $\{z_i\}_{i=1, l-1}$ are the solutions to the finite dimensional variational inequalities

$$\left(V_d^i\right) \quad z_i'(t) + \partial I_K(z_i(t)) \ni 0 \quad \text{a. e.} \quad t \in]0, d[,$$

with the initial conditions

$$(3) \quad z_i(0) = P \left[\left(y_d^u\right)^-(id) \right]$$

for $i=1, 2, \dots, l-1$, where P is the projection of $\mathbb{R}^n$ on K . (This represents the fractional step scheme [1].)

Remark 1. From the variational inequalities $\left(V_d^i\right)_{i=1, l-1}$ we obtain

$$(4) \quad z_i(t) = e^{-t\partial I_K} P \left[\left(y_d^u\right)^-(id) \right],$$

which leads to

$$(5) \quad z_i(d) = e^{-d\partial I_K} P \left[\left(y_d^u\right)^-(id) \right].$$

The relation (2) becomes now

$$(2'') \quad \left(y_d^u\right)^+(id) = P \left[\left(y_d^u\right)^-(id) \right] \quad \text{for} \quad i = \overline{1, l-1},$$

using the equality $e^{-d\partial I_K} p = p$.

Remark 2. For $i=0$ we have considered the differential equation

$$\left(E_d^0\right) \quad y'(t) + F(y(t)) = Bu(t) + f(t) \quad \text{a. e.} \quad t \in]0, d[,$$

with the initial condition

$$(2'') \quad y^*(0) = y_0 \in K.$$

In the following we denote $Bu + f = w$ on $[0, T]$.

The next Lemma will establish a relation between solution y_d^w to the equation

$$(6) \quad \left(y_d^w(t)\right)' + F\left(y_d^w(t)\right) = w(t) \quad \text{a. e.} \quad t \in](i-1)d, id[,$$

with the initial condition

$$(7) \quad \left(y_d^w\right)^+((i-1)d) = P \left[\left(y_d^w\right)^-((i-1)d) \right], \quad i = \overline{2, l},$$

and the solution y_{1d}^w to the corresponding homogeneous equation with the same initial condition.

L e m m a 1. *We have the equality*

$$(8) \quad y_d^w - y_{1d}^w + y_{2d}^w \quad \text{a. e.} \quad t \in](i-1)d, id[,$$

where

$$(9) \quad \left\| \left(y_{2d}^w\right)^-(id) \right\| \leq C \int_{(i-1)d}^{id} \|w(t)\| dt, \quad i = \overline{1, l}.$$

P r o o f. If we subtract the equation satisfied by y_{1d}^w from the equation satisfied by y_d^w and multiply scalar the result by $\operatorname{sgn}(y_d^w - y_{1d}^w)$, taking into account the properties of the elements which appear, and applying Gronwall lemma we get the relations (8) and (9).

We impose the supplementary, nonrestrictive, condition that the homogeneous equation

$$(10) \quad y'(t) + F(y(t)) = 0 \quad \text{a. e.} \quad t \in]0, T[,$$

with the initial condition

$$(11) \quad y(0) = y_0 \in K,$$

admits a unique solution with the property

$$(12) \quad y(t) \in K \quad \text{a. e.} \quad t \in]0, T[.$$

P r o p o s i t i o n 1. *For every $w \in L^2(0, T; \mathbb{R}^n)$ we have the convergence*

$$(13) \quad \lim_{d_n \rightarrow 0} y_{d_n}^w = y^w \quad \text{in} \quad BV([0, T]; \mathbb{R}^n)$$

on a subsequence $\{d_n\}_{n \in \mathbb{N}} \subset \{d\}_{d>0}$.

We take the scalar product of equation (E_d^i) , $i = \overline{0, l-1}$ written for y_d^w , with y_d^w , we employ condition (2''), then we integrate it on the closed interval $[id, t]$, use Gronwall lemma and take $t = (i+1)d$ and we obtain

$$(14) \quad \left\| \left(y_D^w\right)^-((i+1)d) \right\|^2 \leq \left[\left\| \left(y_D^w\right)^-(id) \right\|^2 + \int_0^d \|w(s)\|^2 ds + 2D_2d \right] e^{Dd},$$

for $i=1, 2, \dots, l-1$.

Analogously, the differential equation (H_d^0) leads to

$$(15) \quad \left\| \left(y_d^w \right)^- (d) \right\|^2 \leq \left[\|y_0\|^2 + \int_0^d \|w(s)\|^2 ds + 2D_2 d \right] e^{Dd}, \text{ where } D = 2D_1 + 1.$$

From the inequalities (14) and (15) we get

$$(16) \quad \left\| \left(y_d^w \right)^- ((i+1)d) \right\|^2 \leq \left[\|y_0\|^2 + \int_0^{(i+1)d} \|w(s)\|^2 ds + 2D_2 T \right] e^{DT},$$

therefore

$$(16') \quad \left\| \left(y_d^w \right)^- (id) \right\| \leq C, \quad \text{for } i = \overline{1, l},$$

which implies, using (2''),

$$\left\| \left(y_d^w \right)^+ (id) \right\| \leq C, \quad \text{for } i = \overline{0, l-1},$$

where

$$C^2 = e^{DT} \left(\|y_0\|^2 + 2D_2 T + \|w\|_{L^2(0, T; \mathbb{R}^n)}^2 \right).$$

For every $t \in [0, T]$ there exists $0 \leq i \leq l-1$ so that $t \in [id, (i+1)d[$ and y_d^w satisfies (E_d^i) , i. e.

$$(17) \quad \left(y_d^w(t) \right)' + F \left(y_d^w(t) \right) = w(t) \quad \text{a. e.} \quad t \in]id, (i+1)d[,$$

with the initial condition

$$(18) \quad \left(y_d^w \right)^+ (id) = P \left[\left(y_d^w \right)^- (id) \right].$$

Analogously, we get

$$(19) \quad \|y_d^w\|_{C([0, T]; \mathbb{R}^n)} \leq C_1,$$

where $C_1^2 = e^{DT} (C^2 + 2D_2 T + \|w\|_{L^2(0, T; \mathbb{R}^n)}^2)$ which means that the sequence $\{y_d^w\}_{d>0}$ is bounded in the space $C([0, T^w]; \mathbb{R}^n)$.

We have the relation

$$(20) \quad \begin{aligned} S_d^w &= \sum_{i=1}^{l-1} \left\| \left(y_d^w \right)^- (id) - \left(y_d^w \right)^+ (id) \right\| \leq \\ &\leq \sum_{i=1}^{l-1} \left\| P \left[\left(y_{1d}^w \right)^- (id) \right] - P \left[\left(y_d^w \right)^- (id) \right] \right\| + \sum_{i=1}^{l-1} \left\| \left(y_{2d}^w \right)^- (id) \right\|, \end{aligned}$$

which, in virtue of Lemma 1, becomes

$$(20') \quad S_d^w = C_2 \|w\|_{L^1(0, T; \mathbb{R}^n)}, \quad C_2 = C(l-1),$$

taking into account condition (2'').

By applying Helly theorem, we conclude that on a subsequence $\{d_n\}_{n \in \mathbb{N}} \subset \{d\}_{d>0}$, we have

$$(21) \quad \lim_{d_n \rightarrow 0} y_{d_n}^w(t) = y^*(t) \quad \text{for every } t \in [0, T].$$

We prove that $y^*(t) = y^w(t)$ a.e. $t \in [0, T]$. At first, we establish that

$$(22) \quad y^*(t) \in K \quad \text{a.e. } t \in]0, T[.$$

By Lemma 1 there follows that

$$(23) \quad \begin{aligned} \|y_n(t) - Py_n(t)\| &= \|y_{1n}(t) + y_{2n}(t) - P(y_{1n}(t) + y_{2n}(t))\| \leq \\ &\leq \|Py_{1n}(t) - P(y_{1n}(t) + y_{2n}(t))\| + \|y_{2n}(t)\| \leq \\ &\leq 2\|y_{2n}(t)\| \leq 2C \int_{(i-1)d_n}^t \|w(s)\| ds \leq 2cd_n^{1/2} \|w\|_{L^2(0, T; \mathbb{R}^n)}, \end{aligned}$$

for every $t \in [(i-1)d_n, id_n[$; we denote $y_{d_n}^w = y_n$.

The relation (22) results from (23) letting d_n tend to 0.

Now let $y \in K$ be arbitrary but fixed and let $s \in [id, (i+1)d[$, $t \in [jd, (j+1)d[$, with $i < j$ be two points on the interval $[0, T]$.

From the relation

$$(24) \quad y_n'(t) + F(y_n(t)) = w(t),$$

multiplied with $y_n(t) - y$, there results

$$(24') \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|y_n(t) - y\|^2 + (F(y_n(t)), y_n(t) - y) = \\ = (w(t), y_n(t) - y), \end{aligned}$$

which integrated on the interval $[s, t]$, leads to

$$\begin{aligned}
 (24'') \quad & \frac{1}{2} \|y_n(t) - y\|^2 - \frac{1}{2} \|y_n(s) - y\|^2 = \\
 & = \int_s^t (w(\tau) - F(y_n(\tau)), y_n(\tau) - y) d\tau + \\
 & + \frac{1}{2} \sum_{m=i+1}^j \|y_n^+(md) - y\|^2 - \|y_n^-(md) - y\|^2.
 \end{aligned}$$

We have the inequality

$$\begin{aligned}
 (25) \quad & \frac{1}{2} \|y_n^+(md) - y\|^2 - \|y_n^-(md) - y\|^2 \leq \\
 & \leq (y_n^+(md) - y_n^-(md), y_n^+(md) - y) = \\
 & = \left[P(y_n^-(md)) - y_n^-(md), P(y_n^-(md)) - y \right] = \\
 & = - \left[(I - P)y_n^-(md), P(y_n^-(md)) - Py \right] \leq 0,
 \end{aligned}$$

because $I - P = \lambda(\partial I_k)_\lambda \in \lambda \partial I_K P$ for every $\lambda > 0$. Hence

$$\frac{1}{2} \|y_n(t) - y\|^2 - \frac{1}{2} \|y_n(s) - y\|^2 \leq \int_s^t (w(\tau) - Fy_n(\tau), y_n(\tau) - y) d\tau,$$

for every $0 \leq s \leq t \leq T$. For $d_n \rightarrow 0$ we get

$$(26) \quad \frac{1}{2} \|y^*(t) - y\|^2 - \frac{1}{2} \|y^*(s) - y\|^2 \leq \int_s^t (w(\tau) - Fy(\tau), y(\tau) - y) d\tau$$

for every $0 \leq s \leq t \leq T$.

If in relation (26) we take $y^* = y(s) \in K$ and apply Gronwall lemma we obtain

$$(27) \quad \|y^*(t) - y^*(s)\|^2 \leq e^T \int_s^t \|w(\tau) - Fy(\tau)\|^2 d\tau,$$

for $0 \leq s \leq t \leq T$ and therefore $y^*: [0, T] \rightarrow \mathbb{R}^n$ is absolutely continuous and a.e. differentiable.

The relation (26) implies

$$(26') \quad (y^*(t) - y^*(s), y^*(s) - y) \leq \int_s^t (w(\tau) - Fy^*(\tau), y^*(\tau) - y) d\tau,$$

which divided by $t - s$ and letting $t - s$ tend to 0 becomes

$$(28) \quad (y^{*'}(t), y^*(t) - y) \leq (w(t) - Fy^*(t), y^*(t) - y) \quad \text{a. e.} \quad t \in]0, T[,$$

for all $y \in K$, which means that

$$(29) \quad y^{*'}(t) + (F + \partial I_K)y^*(t) \ni w(t) \quad \text{a. e.} \quad t \in]0, T[.$$

In other words, we have proved that $y^* = y^w$ on $[0, T]$.

Analogously, the following result can be proved.

Proposition 2. *If $w_{d_n} \rightarrow w$ weakly in $L^1(0, T; \mathbb{R}^n)$ when d_n tends to 0, then we have the convergence*

$$(30) \quad \lim_{d_j \rightarrow 0} y_{d_j}^{w_{d_j}} = y^w \quad \text{on} \quad BV([0, T]; \mathbb{R}^n),$$

on a subsequence $\{d_j\}_{j \in \mathbb{N}} \subset \{d_n\}_{n \in \mathbb{N}}$.

Received January 19, 1994

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O METODĂ PENTRU APROXIMAREA SOLUȚIEI UNEI INEGALITĂȚI VARIATIONALE FINIT DIMENSIONALE

(Rezumat)

Se aplică metoda pașilor fracționari (v. [1]) inegalității variaționale dimensionale (V_K) cu condiția inițială (IC).

THE DARBOUX PROBLEM FOR RANDOM HYPERBOLIC MULTIVALUED EQUATIONS

BY

GEORGETA TEODORU and VLADIMIR DRĂGAN

1. Introduction. In this paper we shall study the Darboux problem for a multivalued hyperbolic equation

$$(1.1) \quad \frac{\partial^2 z}{\partial x \partial y} \in \varphi(\omega, x, y, z),$$

where ω is a random parameter and φ is a given set-valued mapping. Using the random fixed point theorem for set-valued mappings of Nowak [11], we prove an existence theorem of random absolutely continuous solution which depends measurably on a random parameter.

We also consider the Darboux problem associated with the multivalued hyperbolic equation

$$(1.2) \quad \frac{\partial^2 z}{\partial x \partial y} \in \varphi \left[x, y, z, \int_0^x \int_0^y \psi(x, y, p, q, z(p, q)) dp dq \right],$$

for which we prove the existence of a global solution using the fixed point principle for multifunctions of Covitz and Nadler [2], [15], [16].

Finally, for the random version of Eq. (1.2) given by

$$(1.3) \quad \frac{\partial^2 z}{\partial x \partial y} \in \varphi \left[\omega, x, y, z, \int_0^x \int_0^y \psi(x, y, p, q, z(p, q)) dp dq \right],$$

we give an existence theorem for the Darboux problem, also using the random fixed point theorem for set-valued mappings of Nowak [11].

These theorems are random analogous or analogous of a result obtained by Teodoru [16].

2. Preliminaries. Denote by (Ω, A, P) a complete probability space; $D=[0, a] \times [0, b]$ is a compact rectangle with the corresponding σ -algebra $\mathcal{L}_D$; $CL(X)$ is the class of nonempty closed subsets of metric space X and B_X denotes the Borel σ -algebra on X . By $A \times B_X$ we denote the product σ -algebra on $\Omega \times X$.

The Hausdorff-Pompeiu [14] metric on $CL(X)$, $H: CL(X) \rightarrow \mathbb{R}_+$, is given by $H(A, B) = d_H(A, B) = \max \left\{ \sup \{ d(a, B) : a \in A \}, \sup \{ d(b, A) : b \in B \} \right\} = \max \{ d^*(A, B), d^*(B, A) \}$, where $A, B \in CL(X)$ and

$$d(a, B) = \inf \{ d(a, b) : b \in B \},$$

d being the metric on X .

Let X, Y be two nonempty sets. A multifunction (set-valued mapping (map)) ϕ from X into Y (briefly, $\phi: X \rightarrow 2^Y$) is a function from X into the family of all nonempty subsets of Y .

Let $(X, d), (Y, \rho)$ be two metric spaces. A multifunction $\phi: X \rightarrow 2^Y$ is said to be Lipschitzian if there exists a real number $L > 0$ (Lipschitz constant) such that

$$\rho_H(\phi(x), \phi(y)) \leq L d(x, y)$$

for all $x, y \in X$, ρ_H being the Hausdorff-Pompeiu metric induced by the metric ρ on set of all subsets of Y . If $L < 1$, we say that ϕ is a multivalued contraction [7], [8]. We recall the fixed point principle for multivalued contractions of Covitz and Nadler [2], [15], [16], stating it as

Theorem 2.1 (Covitz-Nadler, [5]). *If (X, ρ) is a complete space and $F: X \rightarrow CL(X)$ a set-valued mapping, i.e. such that $H(F(x), F(y)) \leq k \rho(x, y)$, for every $x, y \in X \times X$ with any $k \in]0, 1[$, then there exists $x \in X$ such that $x \in F(x)$.*

Denote by $C(D; \mathbb{R}^n)$ the space of all continuous functions $z: D \rightarrow \mathbb{R}^n$ with the norm

$$\|z\| = \max \{ |z(x, y)| : (x, y) \in D \},$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$. For $A \subset \mathbb{R}^n$,

$$|A| = \sup \{ |a| : a \in A \}.$$

By $L^1(D; \mathbb{R}^n)$ we mean the space of all integrable functions $f: D \rightarrow \mathbb{R}^n$ endowed with the norm

$$\|f\| = \int \int_D |f(x, y)| dx dy$$

(more precisely, the elements of $L^1(D; \mathbb{R}^n)$ are classes of equivalent functions). It is well known that $C(D; \mathbb{R}^n)$ and $L^1(D; \mathbb{R}^n)$ are separable Banach spaces.

Let $\phi: \Omega \rightarrow CL(\mathbb{R}^n)$ be a set-valued map. By the *graph* of ϕ we mean

$$\text{gr}\phi = \{(\omega, x) \in \Omega \times X: x \in \phi(\omega)\}, \quad X = \mathbb{R}^n.$$

We call ϕ *measurable* [11] if for each open set $G \subset X = \mathbb{R}^n$

$$\{\omega \in \Omega: \phi(\omega) \cap G \neq \emptyset\} \in \mathcal{A}.$$

A set-valued map $\phi: X \rightarrow CL(Y)$ where Y is a metric space, is *closed* if its graph is closed in $X \times Y$.

Let X be a metric space and $\phi: \Omega \times X \rightarrow CL(X)$ a set-valued map. A function $\xi: \Omega \rightarrow X$ is a *random fixed point* of ϕ if it is measurable and $\xi(\omega) \in \phi(\omega, \xi(\omega))$ P -almost surely.

Random fixed point theorems for set-valued mappings were given by several authors. We recall Nowak's theorem [11], which is a random version of the Nadler fixed point theorem.

Theorem 2.2 (A. Nowak [11]). *Let $(X, \|\cdot\|)$ be a separable Banach space and $\phi: \Omega \times X \rightarrow CL(X)$ a set-valued map such that for each $x \in X$, $\phi(\cdot, x)$ is measurable, and for each $\omega \in \Omega$ there exists a norm $\|\cdot\|_\omega$ on X equivalent to $\|\cdot\|$ and a constant $\lambda_\omega < 1$ such that*

$$H_\omega(\phi(\omega, x), \phi(\omega, y)) \leq \lambda_\omega \|x - y\|, \quad x, y \in X,$$

where H_ω is the Hausdorff-Pompeiu metric on $CL(X)$ induced by $\|\cdot\|_\omega$. Then ϕ has a random fixed point.

As regards $AC([\alpha_1, \alpha_2], \mathbb{R}^n)$ and $C^*(D; \mathbb{R}^n)$ — the space of all absolutely continuous functions in the Carathéodory sense on D , see [16], and [1, § 565 - § 568].

3. Results. We consider the Darboux problem for the random multivalued equation

$$(3.1) \quad \frac{\partial^2 z}{\partial x \partial y} \in \varphi(\omega, x, y, z),$$

with the conditions

$$(3.2) \quad \begin{aligned} z(\omega, x, 0) &= \sigma(\omega, x), \quad x \in [0, a], \quad z(\omega, 0, y) = \tau(\omega, y), \\ y &\in [0, b], \quad \sigma(\omega, 0) = \tau(\omega, 0), \end{aligned}$$

where $\varphi: \Omega \times D \times \mathbb{R}^n \rightarrow CL(\mathbb{R}^n)$ and $\sigma: \Omega \times [0, a] \rightarrow \mathbb{R}^n$, $\tau: \Omega \times [0, b] \rightarrow \mathbb{R}^n$ are given mappings.

A function $\xi: \Omega \times D \rightarrow \mathbb{R}^n$ is called a *random solution* of the Darboux problem (3.1)-(3.2) if it is measurable in ω , absolutely continuous in (x, y) (in

the Carathéodory sense [1, § 565-§ 568]), $\xi \in C^*(D; \mathbb{R}^n)$, and for P -almost all $\omega \in \Omega$ satisfies (3.1), i.e.

$$\frac{\partial^2 \xi}{\partial x \partial y}(\omega, x, y) \in \varphi(\omega, x, y, \xi(\omega, x, y))$$

almost everywhere (abbreviated to a.e.) in D and the conditions (3.2), i.e.

$$\xi(\omega, x, 0) = \sigma(\omega, x), \quad \xi(\omega, 0, y) = \tau(\omega, y), \quad x \in [0, a], \quad y \in [0, b].$$

By I we denote the continuous linear operator $I: L^1(D; \mathbb{R}^n) \rightarrow C(D; \mathbb{R}^n)$ defined as

$$(3.3) \quad I[f](x, y) = \int_0^x \int_0^y f(u, v) du dv, \quad f \in L^1(D; \mathbb{R}^n).$$

The following existence theorem for a random solution of the problem (3.1)-(3.2) is the random analogous of the result by Teodoru [16].

Theorem 3.1. Let $\varphi: \Omega \times D \times \mathbb{R}^n \rightarrow 2^{\mathbb{R}^n}$ be a multivalued mapping which satisfies the following hypotheses:

(H₁) $\varphi(\omega, x, y, z)$ is closed for each $(\omega, x, y, z) \in \Omega \times D \times \mathbb{R}^n$;

(H₂) $\varphi(\cdot, \cdot, \cdot, z)$ is $A \times \mathcal{L}_D$ -measurable for each $z \in \mathbb{R}^n$;

(H₃) For each $\omega \in \Omega$ there exists an integrable function $\alpha_\omega: D \rightarrow \mathbb{R}_+$, $\alpha_\omega \in L_1(D; \mathbb{R}_+)$, such that

$$|\varphi(\omega, x, y, z)| \leq \alpha_\omega(x, y),$$

for all $(x, y) \in D, z \in \mathbb{R}^n$;

(H₄) $\varphi(\omega, x, y, \cdot)$ is Lipschitzian in the last variable, i.e. for each $\omega \in \Omega$ there exists a function $k_\omega: D \rightarrow \mathbb{R}_+$, $k_\omega \in L^2(D; \mathbb{R}_+)$ such that

$$H(\varphi(\omega, x, y, z_1), \varphi(\omega, x, y, z_2)) \leq k_\omega(x, y) |z_1 - z_2|,$$

for all $(x, y) \in D, z_1, z_2 \in \mathbb{R}^n$, where H is the Hausdorff-Pompeiu metric on $CL(\mathbb{R}^n)$, and let k_0 be a constant such that

$$k_0 \geq \left[\int_s^a \int_t^b k_\omega^2(u, v) du dv \right]^{1/2}, \quad 0 \leq s \leq a, \quad 0 \leq t \leq b;$$

(H₅) $\beta > 0$ is a constant such that $k_0/2\beta = L < 1$;

(H₆) The functions $\sigma(\cdot, x), \tau(\cdot, y)$ are A -measurable for each $x \in [0, a], y \in [0, b], \sigma(\omega, \cdot) \in AC([0, a]; \mathbb{R}^n), \tau(\omega, \cdot) \in AC([0, b]; \mathbb{R}^n)$ for each $\omega \in \Omega, \sigma(\omega, 0) = \tau(\omega, 0)$ for each $\omega \in \Omega$.

Then the Darboux problem (3.1)-(3.2) has at least one random solution.

P r o o f. From (H_6) it results that the function $\alpha: \Omega \times D \rightarrow \mathbb{R}^n$, defined by

$$\alpha(\omega, x, y) = \sigma(\omega, x) + \tau(\omega, y) - \sigma(\omega, 0), \quad (x, y) \in D, \quad \omega \in \Omega,$$

is absolutely continuous in the Carathéodory sense on D , [1, § 565-§ 568], $\alpha \in C^*(D; \mathbb{R}^n)$, for each $\omega \in \Omega$.

We consider the multifunction $\phi_\alpha: \Omega \times L^1(D; \mathbb{R}^n) \rightarrow 2^{L^1(D; \mathbb{R}^n)}$ defined by

$$(3.4) \quad \phi_\alpha(\omega, f) = \{g \in L^1(D; \mathbb{R}^n) : g(x, y) \in \varphi(\omega, x, y, \alpha(\omega, x, y) + I[f](x, y)) \text{ a.e. in } D\}$$

for all $\omega \in \Omega, f \in L^1(D; \mathbb{R}^n)$.

It is not difficult to see that if $\xi: \Omega \rightarrow L^1(D; \mathbb{R}^n)$ is a random fixed point of ϕ_α , then the function

$$(3.5) \quad \xi(\omega, x, y) = \alpha(\omega, x, y) + I[\xi(\omega)](x, y), \quad \omega \in \Omega, \quad (x, y) \in D,$$

where I is given by (3.3), is a random solution of the Darboux problem (3.1) - (3.2).

We shall prove that ϕ_α given by (3.4) satisfies the assumption of Theorem 2.2. Fix $\omega \in \Omega$ and $f \in L^1(D; \mathbb{R}^n)$. From (H_2) and (H_4) it follows that the set-valued mapp

$$(x, y) \rightarrow \varphi(\omega, x, y, \alpha(\omega, x, y) + I[f](x, y))$$

is $\mathcal{L}_D$ -measurable, according to Proposition 1.2, [9]. Hence, by Proposition 2.1, [10], there exists a measurable function $g: D \rightarrow \mathbb{R}^n$ such that

$$g(x, y) \in \varphi(\omega, x, y, \alpha(\omega, x, y) + I[f](x, y)), \quad \text{i.e. } g \in \phi_\alpha(\omega, f),$$

and

$$|g(x, y)| \leq \rho(\theta, \varphi(\omega, x, y, \alpha(\omega, x, y) + I[f](x, y))) + 1, \quad (x, y) \in D,$$

where ρ is a metric in $\mathbb{R}^n$, and θ is the zero-vector in $\mathbb{R}^n$.

According to (H_3) , g is integrable, $g \in L^1(D; \mathbb{R}^n)$. Hence $\phi_\alpha(\omega, f) \neq \emptyset$.

Consider now the function $r_{\alpha, f}: \Omega \times L^1(D; \mathbb{R}^n) \rightarrow \mathbb{R}$, defined by

$$r_{\alpha, f}(\omega, g) = \int \int_D \rho(g(x, y), \varphi(\omega, x, y, \alpha(\omega, x, y) + I[f](x, y))) dx dy,$$

for all $\omega \in \Omega, g \in L^1(D; \mathbb{R}^n)$.

It is immediate that

$$\phi_\alpha(\omega, f) = \{g \in L^1(D; \mathbb{R}^n) : r_{\alpha, f}(\omega, g) = 0\}.$$

We shall prove that $r_{\alpha, f}$ is well defined, measurable with respect to ω , and continuous with respect to g . To this end, let us remark that, by (H_2) , (H_4) and Proposition 2 of [6], the multifunction

$$(\omega, x, y) \rightarrow \varphi(\omega, x, y, \alpha(\omega, x, y) + I[f](x, y))$$

is $A \times \mathcal{L}_D$ -measurable. Therefore, by Lemma 2.1 of [13], for every $g \in L^1(D; \mathbb{R}^n)$ the real function

$$(\omega, x, y) \rightarrow \rho(g(x, y), \varphi(\omega, x, y, \alpha(\omega, x, y) + I[f](x, y)))$$

is $A \times \mathcal{L}_D$ -measurable. On the other hand it follows from (H_3) that for every $\omega \in \Omega$, the real function

$$(x, y) \rightarrow \rho(g(x, y), \varphi(\omega, x, y, \alpha + I[f](x, y)))$$

belongs to $L^1(D; \mathbb{R})$. This implies that $r_{\alpha, f}$ is well defined and, by Fubini's theorem [3], [12], it is also measurable with respect to ω .

Fix $\omega \in \Omega$. Since for every $g_1, g_2 \in L^1(D; \mathbb{R}^n)$ we have

$$\begin{aligned} & |r_{\alpha, f}(\omega, g_1) - r_{\alpha, f}(\omega, g_2)| \leq \\ & \leq \int \int \left| \rho(g_1(x, y), \varphi(\omega, x, y, \alpha + I[f](x, y))) - \rho(g_2(x, y), \varphi(\omega, x, y, \alpha + I[f](x, y))) \right| dx dy \leq \int \int |g_1(x, y) - g_2(x, y)| dx dy = \|g_1 - g_2\|_{L^1(D; \mathbb{R}^n)}, \end{aligned}$$

the function $r_{\alpha, f}$ is continuous with respect to g . Hence, for each $\omega \in \Omega$, the set $\phi_{\alpha}(\omega, f)$ is a closed subset of $L^1(D; \mathbb{R}^n)$. Being measurable in the first variable and continuous in the second, g is product $A \times B_{L^1(D; \mathbb{R}^n)}$ -measurable, where $B_{L^1(D; \mathbb{R}^n)}$ denotes the Borel σ -algebra of $L^1(D; \mathbb{R}^n)$. The last assertion implies that

$$\begin{aligned} \text{gr}(\phi_{\alpha}(\cdot, f)) &= \{(\omega, g) \in \Omega \times L^1(D; \mathbb{R}^n) : r_{\alpha, f}(\omega, g) = 0\} \in \\ &\in A \times B_{L^1(D; \mathbb{R}^n)}. \end{aligned}$$

Therefore, (Ω, A, P) being complete, we obtain from [5] that the multifunction $\omega \rightarrow \phi_{\alpha}(\omega, f)$ is measurable. For every $\omega \in \Omega$ consider on $L^1(D; \mathbb{R}^n)$ the norm (equivalent to the usual one) [16]

$$\|f\|_{\omega} = \int \int e^{-\beta(u+v)} |f(u, v)| du dv, \quad f \in L^1(D; \mathbb{R}^n)$$

and denote by ρ_{ω} the metric induced by $\|\cdot\|_{\omega}$.

Let us prove that, for every fixed $\omega \in \Omega$, the multifunction $\phi_{\alpha}(\omega, \cdot)$ is a multivalued contraction on $(L^1(D; \mathbb{R}^n), \rho_{\omega})$ with the Lipschitz constant $k_0/2\beta = L < 1$.

If $f_1, f_2 \in L^1(D; \mathbb{R}^n)$ $g \in \phi_\alpha(\omega, f_1)$ and $\varepsilon > 0$, according to Proposition 2.1 of [10], there exists a measurable function $h: D \rightarrow \mathbb{R}^n$ such that

$$h(x, y) \in \varphi(\omega, x, y, \alpha + I[f_2](x, y)), \text{ i.e. } h \in \phi_\alpha(\omega, f_2),$$

and

$$|h(x, y) - g(x, y)| \leq \rho(g(x, y), \varphi(\omega, x, y, \alpha + I[f_2](x, y))) + \varepsilon,$$

for all $(x, y) \in D$. Then $h \in L^1(D; \mathbb{R}^n)$ and we have from (H_4) that

$$\begin{aligned} \|g - h\|_\omega &= \int_D \int e^{-\beta(u+v)} |g(u, v) - h(u, v)| \, du \, dv \leq \\ &\leq \int_D \int e^{-\beta(u+v)} \rho(g(u, v), \varphi(\omega, u, v, \alpha(\omega, u, v) + I[f_2](u, v))) \, du \, dv + \varepsilon ab \leq \\ &\leq \int_D \int \left[e^{-\beta(u+v)} k_\omega(u, v) \int_0^u \int_0^v |f_2(s, t) - f_1(s, t)| \, ds \, dt \right] \, du \, dv + \varepsilon ab. \end{aligned}$$

By interchanging the order of integration, we obtain

$$\|g - h\|_\omega \leq \int_D \int |f_2(s, t) - f_1(s, t)| \left[\int_s^a \int_t^b k_\omega(u, v) e^{-\beta(u+v)} \, ds \, dv \right] \, ds \, dt + \varepsilon ab.$$

Using the Hölder inequality [16], it result that

$$\begin{aligned} \int_s^a \int_t^b k_\omega(u, v) e^{-\beta(u+v)} \, du \, dv &\leq \left[\int_s^a \int_t^b k_\omega^2(u, v) \, du \, dv \right]^{1/2} \left[\int_s^a \int_t^b e^{-2\beta(u+v)} \, du \, dv \right]^{1/2} \leq \\ &\leq k_0 \left[\frac{e^{-2\beta s} - e^{-2\beta a}}{2\beta} \frac{e^{-2\beta t} - e^{-2\beta b}}{2\beta} \right]^{1/2} \leq k_0 \left[\frac{e^{-2\beta(s+t)}}{4\beta^2} \right]^{1/2} = \\ &= \frac{k_0}{2\beta} e^{-\beta(s+t)} = L e^{-\beta(s+t)}. \end{aligned}$$

Therefore

$$\begin{aligned} \|g - h\|_\omega &\leq L \int_D \int e^{-\beta(s+t)} |f_2(s, t) - f_1(s, t)| \, ds \, dt + \varepsilon ab = \\ &= L \|f_1 - f_2\|_\omega + \varepsilon ab. \end{aligned}$$

This implies that $\rho_\omega(g, \phi_\alpha(\omega, f)) \leq L \|f_1 - f_2\|_\omega$. Hence

$$\rho_{\omega}^*(\phi_{\alpha}(\omega, f_1), \phi_{\alpha}(\omega, f_2)) \leq L \|f_1 - f_2\|_{\omega}$$

and, interchanging the roles of f_1 and f_2 we get

$$\rho_{\omega}^*(\phi_{\alpha}(\omega, f_2), \phi_{\alpha}(\omega, f_1)) \leq L \|f_1 - f_2\|_{\omega}.$$

We have

$$H_{\omega}(\phi_{\alpha}(\omega, f_1), \phi_{\alpha}(\omega, f_2)) \leq L \|f_1 - f_2\|_{\omega},$$

for all $f_1, f_2 \in L^1(D; \mathbb{R}^n)$, $L < 1$, where H_{ω} is the Hausdorff-Pompeiu metric on $CL(L^1(D; \mathbb{R}^n))$ induced by the norm $\|\cdot\|_{\omega}$, i.e. $\phi_{\alpha}(\omega, \cdot)$ given by (3.4) is a multivalued contraction. It follows from Theorem 2.2 that ϕ_{α} has a random fixed point of the form (3.5) which is a random solution of (3.1)–(3.2). This completes the proof.

Consider now the Darboux problem associated with the multivalued hyperbolic equation

$$(3.6) \quad \frac{\partial^2 z}{\partial x \partial y} \in \varphi \left[x, y, z, \int_0^x \int_0^y \psi(x, y, p, q, z(p, q)) dp dq \right],$$

where $\psi: D \times D \times \mathbb{R}^n \rightarrow \mathbb{R}^m$, $\varphi: D \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow 2^{\mathbb{R}^n}$ are given mappings with the conditions

$$(3.7) \quad z(x, 0) = \sigma(x), \quad 0 \leq x \leq a, \quad z(0, y) = \tau(y), \quad 0 \leq y \leq a, \quad \sigma(0) = \tau(0),$$

where $\sigma \in AC([0, a]; \mathbb{R}^n)$; $\tau \in AC([0, b]; \mathbb{R}^n)$.

This problem consists in determining the *solution* for (3.6) defined as an absolutely continuous (in the Carathéodory sense) function $z: D \rightarrow \mathbb{R}^n$, $z \in C^*(D; \mathbb{R}^n)$, which satisfies (3.6) a.e. $(x, y) \in D$ and conditions (3.7).

We shall prove an existence theorem of the Darboux problem (3.6)–(3.7) which is similar to [16], also using the fixed point principle of Covitz and Nadler for multifunctions.

Theorem 3.2. Assume the following hypotheses:

(H₁') $\varphi(x, y, z, w)$ is closed for each $(x, y, z, w) \in D \times \mathbb{R}^n \times \mathbb{R}^m$;

(H₂') $\varphi(\cdot, \cdot, z, w)$ is measurable for each $(z, w) \in \mathbb{R}^n \times \mathbb{R}^m$;

(H₃') $\varphi(x, y, \cdot, \cdot)$ is Lipschitzian in the last two variables, i.e. there exists a function $k: D \rightarrow \mathbb{R}_+$, $k \in L^2(D; \mathbb{R}_+)$ such that

$$h(\varphi(x, y, z_1, w_1), \varphi(x, y, z_2, w_2)) \leq k(x, y)(|z_1 - z_2| + |w_1 - w_2|),$$

for all $(z_1, w_1), (z_2, w_2) \in \mathbb{R}^n \times \mathbb{R}^m$; h is the Hausdorff-Pompeiu metric on $CL(\mathbb{R}^n)$. Also, $\psi(x, y, p, q, \cdot)$ is Lipschitzian in the last variable, i.e. there

exists a function $\mu: D \times D \rightarrow \mathbb{R}_+$, $\mu \in L^1(D \times D; \mathbb{R}_+)$, such that

$$|\psi(x, y, p, q, z_1) - \psi(x, y, p, q, z_2)| \leq \mu(x, y, p, q) |z_1 - z_2|$$

for all $(x, y) \in D$, $(p, q) \in D$, $z_1, z_2 \in \mathbb{R}^n$.

Let $k_0 > 0$ be a constant with

$$k_0 \geq \left[\int_s^b \int_t^b k^2(u, v) \left[1 + \int_0^u \int_0^v \mu(u, v, p, q) dp dq \right]^2 du dv \right]^{1/2},$$

$$0 \leq s \leq a, \quad 0 \leq t \leq b;$$

(H₄') There is a function $f \in L^1(D; \mathbb{R}_+)$ such that for $v \in \varphi(x, y, z, w)$,

$(z, w) \in \mathbb{R}^n \times \mathbb{R}^m$, $|v| \leq f(x, y)$ a.e. for $(x, y) \in D$;

(H₅') $\beta > 0$ is a constant such that $k_0/2\beta = L < 1$;

(H₆') The functions $\sigma \in AC([0, a]; \mathbb{R}^n)$, $\tau \in AC([0, b]; \mathbb{R}^n)$ satisfy $\sigma(0) = \tau(0)$.

Then there exists a solution of the Darboux problem (3.6)–(3.7) on D .

P r o o f. We first renorm $L^1(D; \mathbb{R}^n)$ with an equivalent norm [16] by

$$\|f\| = \int_D e^{-\beta(u+v)} |f(u, v)| du dv, \quad f \in L^1(D; \mathbb{R}^n).$$

It results from (H₆') that the function $\alpha: D \rightarrow \mathbb{R}^n$, defined by

$$(3.8) \quad \alpha(x, y) = \sigma(x) + \tau(y) - \sigma(0), \quad (x, y) \in D,$$

is absolutely continuous in the Carathéodory sense on D , $\alpha \in C^*(D; \mathbb{R}^n)$, [16], [1, § 565–§ 568].

Consider the multifunction $\phi_\alpha: L^1(D; \mathbb{R}^n) \rightarrow 2^{L^1(D; \mathbb{R}^n)}$, defined by

$$(3.9) \quad \phi_\alpha(f) = \left\{ g \in L^1(D; \mathbb{R}^n) : g(x, y) \in \varphi \left[x, y, \alpha(x, y) + \int_0^x \int_0^y f(s, t) ds dt, \right. \right. \\ \left. \left. \int_0^x \int_0^y \psi \left[\delta, r, x, y, \alpha(\delta, r) + \int_0^\delta \int_0^r f(s, t) ds dt \right] d\delta dr \right] \text{ a.e. in } D \right\}.$$

Obviously, ϕ_α has closed nonempty values. Let $f_1, f_2 \in L^1(D; \mathbb{R}^n)$, $g \in \phi_\alpha(f_1)$ and $\varepsilon > 0$. Then, by Proposition 2.1 of [10], there exists a measurable function $h \in L^1(D; \mathbb{R}^n)$, $h \in \phi_\alpha(f_2)$ such that

$$\begin{aligned}
|h(x, y) - g(x, y)| &\leq \rho \left[g(x, y), \varphi \left[x, y, \alpha(x, y) + \int_0^x \int_0^y f_2(s, t) ds dt, \right. \right. \\
&\quad \left. \left. \int_0^x \int_0^y \psi \left[p, q, x, y, \alpha(\delta, r) + \int_0^\delta \int_0^r f_2(s, t) ds dt \right] dp dq \right] \right] + \varepsilon \leq \\
&\leq k(x, y) \int_0^x \int_0^y |f_1(s, t) - f_2(s, t)| ds dt \left[1 + \int_0^x \int_0^y \mu(p, q, x, y) dp dq \right] + \varepsilon.
\end{aligned}$$

According to (H₃') we have

$$\begin{aligned}
\|h - g\| &= \int_D \int e^{-\beta(u+v)} |h(u, v) - g(u, v)| du dv \leq \\
&\leq \int_D \int e^{-\beta(u+v)} k(u, v) \left[\int_0^u \int_0^v |f_1(s, t) - f_2(s, t)| ds dt \right] \cdot \\
&\quad \cdot \left[1 + \int_0^u \int_0^v \mu(p, q, u, v) dp dq \right] du dv + \varepsilon ab \leq \\
&\leq \frac{k_0}{2\beta} \int_D \int |f_1(s, t) - f_2(s, t)| e^{-\beta(s+t)} ds dt + \varepsilon ab = \\
&= \frac{k_0}{2\beta} \|f_1 - f_2\| + \varepsilon ab = L \|f_1 - f_2\| + \varepsilon ab.
\end{aligned}$$

This implies that $\rho(g, \phi_\alpha(\omega, f)) \leq L \|f_1 - f_2\|$. Hence

$$\rho^*(\phi_\alpha(f_1), \phi_\alpha(f_2)) \leq L \|f_1 - f_2\|.$$

Interchanging the roles of f_1 and f_2 we obtain

$$\rho^*(\phi_\alpha(f_2), \phi_\alpha(f_1)) \leq L \|f_1 - f_2\|.$$

We have

$$H(\phi_\alpha(f_1), \phi_\alpha(f_2)) \leq L \|f_1 - f_2\|$$

for all $f_1, f_2 \in L^1(D; \mathbb{R}^n)$, $L < 1$, where H is the Hausdorff (generalized)

pseudometric $h\|\cdot\|$ on $CL(L^1(D; \mathbb{R}^n))$ induced by the norm $\|\cdot\|$, [2], [4], [16], hence ϕ_α is a multivalued contraction. Finally, we apply the contraction principle of C o v i t z and N a d l e r (Theorem 2.2) to ϕ_α to obtain a function $f \in L^1(D; \mathbb{R}^n)$ such that $f \in \phi_\alpha(f)$.

Let us consider the function $z: D \rightarrow \mathbb{R}^n$ defined by

$$(3.10) \quad z(x, y) = \alpha(x, y) + \int_0^x \int_0^y \bar{f}(p, q) dp dq, \quad (x, y) \in D,$$

where $\bar{f}$ is the fixed point. Then the function (3.10) satisfies the conditions: $z \in C^*(D; \mathbb{R}^n)$,

$$(3.11) \quad \frac{\partial^2 z}{\partial x \partial y} = \bar{f}(x, y) \in \varphi \left[x, y, z(x, y), \int_0^x \int_0^y \psi(p, q, x, y, z(p, q)) dp dq \right],$$

a.e. $(x, y) \in D$

and, in view of (3.8),

$$(3.12) \quad \begin{aligned} z(x, 0) &= \alpha(x, 0) = \sigma(x), \quad 0 \leq x \leq a, \\ z(0, y) &= \alpha(0, y) = \tau(y), \quad 0 \leq y \leq b. \end{aligned}$$

Thus z given by (3.10) is the desired solution of (3.6)–(3.7) on the entire rectangle D . The proof of Theorem 3.2 is now complete.

Finally, we consider the random version of (3.6)–(3.7), hence the Darboux problem

$$(3.13) \quad \frac{\partial^2 z}{\partial x \partial y}(\omega, x, y) \in \varphi \left[\omega, x, y, z, \int_0^x \int_0^y \psi(p, q, x, y, z(p, q)) dp dq \right],$$

$$(3.14) \quad z(\omega, x, 0) = \sigma(\omega, x), \quad z(\omega, 0, y) = \tau(\omega, y), \quad \sigma(\omega, 0) = \tau(\omega, 0),$$

where $\varphi: \Omega \times D \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow 2^{\mathbb{R}^n}$, $\psi: D \times D \times \mathbb{R}^n \rightarrow \mathbb{R}^m$, $\sigma: \Omega \times [0, a] \rightarrow \mathbb{R}^n$, $\tau: \Omega \times [0, b] \rightarrow \mathbb{R}^n$ are given mappings.

A random solution of this problem is defined similarly to the one of (3.1)–(3.2).

Theorem 3.3. Assume the following hypotheses:

(H₁'') $\varphi(\omega, x, y, z, w)$ is closed for each $(\omega, x, y, z, w) \in \Omega \times D \times \mathbb{R}^n \times \mathbb{R}^m$;

(H₂'') $\varphi(\cdot, \cdot, \cdot, z, w)$ is $A \times \mathcal{L}_D$ -measurable for each $(z, w) \in \mathbb{R}^n \times \mathbb{R}^m$;

(H₃'') For each $\omega \in \Omega$ there exists an integrable function $\alpha_\omega: D \rightarrow \mathbb{R}_+$,

$\alpha \in L^1(D; \mathbb{R}^n)$, such that

$$|\varphi(\omega, x, y, z, w)| \leq \alpha_\omega(x, y)$$

for all $(x, y) \in D, (z, w) \in \mathbb{R}^n \times \mathbb{R}^m$;

(H₄'') For each $\omega \in \Omega$, $\varphi(\omega, x, y, \cdot, \cdot)$ is Lipschitzian in the last two variables, i.e. there exists a function $k_\omega: D \rightarrow \mathbb{R}_+$, $k_\omega \in L^2(D; \mathbb{R}_+)$ such that

$$H(\varphi(\omega, x, y, z_1, w_1), \varphi(\omega, x, y, z_2, w_2)) \leq k_\omega(|z_1 - z_2| + |w_1 - w_2|),$$

for all $(z_1, w_1), (z_2, w_2) \in \mathbb{R}^n \times \mathbb{R}^m$; H is the Hausdorff-Pompeiu metric on $CL(\mathbb{R}^n)$.

Also, $\psi(p, q, x, y, \cdot)$ is Lipschitzian in the last variable, i.e. there exists a function $\mu_\omega: D \times D \rightarrow \mathbb{R}_+$, $\mu_\omega \in L^1(D \times D; \mathbb{R}_+)$, such that

$$|\psi(p, q, x, y, z_1) - \psi(p, q, x, y, z_2)| \leq \mu_\omega(p, q, x, y) |z_1 - z_2|,$$

for all $(x, y) \in D, (p, q) \in D, z_1, z_2 \in \mathbb{R}^n$.

Let k_ω^0 be a constant such that

$$k_\omega^0 \geq \left[\int_s^a \int_t^b k_\omega^2(\xi, \eta) \left(1 + \int_0^\xi \int_0^\eta \mu_\omega(p, q, \xi, \eta) dp dq \right)^2 d\xi d\eta \right]^{1/2},$$

$$0 \leq s \leq a, \quad 0 \leq t \leq b;$$

(H₅'') $\beta > 0$ is a constant such that $k_\omega^0/2\beta = L < 1$;

(H₆'') The function $\sigma(\cdot, x), \tau(\cdot, y)$ are A -measurable for each $x \in [0, a], y \in [0, b], \sigma(\omega, \cdot) \in AC([0, a]; \mathbb{R}^n), \tau(\omega, \cdot) \in AC([0, b]; \mathbb{R}^n)$ for each $\omega \in \Omega$, $\sigma(\omega, 0) = \tau(\omega, 0)$ for each $\omega \in \Omega$.

Then the Darboux problem (3.13)–(3.14) has at least one random solution.

The proof is similar to one of Theorem 3.1, therefore we omit it.

Received July 4, 1994

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PROBLEMA LUI DARBOUX PENTRU ECUAȚII MULTIVOCE HIPERBOLICE ALEATOARE

(Rezumat)

Se consideră problema lui Darboux pentru ecuațiile multivoce hiperbolice (3.1), (3.6) și (3.13), unde $\omega \in \Omega$ este un parametru aleator, $(x, y) \in D = [0, a] \times [0, b] \subset \mathbb{R}^2$, $(p, q) \in D$, $z \in \mathbb{R}^n$, φ este aplicație multiunivocă, $\varphi: \Omega \times D \times \mathbb{R}^n \rightarrow 2^{\mathbb{R}^n}$, $\varphi: D \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow 2^{\mathbb{R}^n}$, $\varphi: \Omega \times D \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow 2^{\mathbb{R}^n}$, $\psi: D \times D \times \mathbb{R}^n \rightarrow \mathbb{R}^m$.

Utilizând teorema de punct fix aleator a lui Nowak se demonstrează două teoreme de existență a unei soluții aleatoare a problemei lui Darboux pentru prima și a treia ecuație care depinde măsurabil de parametrul aleator ω . Pentru a doua ecuație se demonstrează o teoremă de existență a unei soluții continue în sens Carathéodory pe baza principiului de punct

fix pentru contracții multivoce al lui C o v i t z și N a d l e r. Toate trei teoremele sunt similare rezultatului obținut de T e o d o r u [16].

RELATIVISTIC MODELS FOR NEWTONIAN CONSERVATIVE SYSTEMS

BY

C. I. BORȘ

1. In this paper we will take into consideration some conservative models of the Newtonian mechanics in order to establish connections with the theory of relativity.

We will use inertial frames. We assume the summation convention over the repeated indices. We will use the following results [3], [4]:

- i) Let $F \in C^2$ as function of the variables $q^h, \dot{q}^h, t (h=1, n)$.
- ii) The Euler-Lagrange equations which derive from the variational problem

$$(1) \quad \delta \int F(q, \dot{q}, t) dt = 0,$$

are given by

$$(2) \quad \frac{d}{dt} \left\{ \frac{\partial F}{\partial \dot{q}^h} \right\} - \frac{\partial F}{\partial q^h} = 0.$$

If F does not depend explicitly on t , we have the integral

$$(3) \quad F - \frac{\partial F}{\partial \dot{q}^h} \dot{q}^h = A,$$

where A is a constant determined by the initial conditions.

Further on we will take into consideration the following models:

- i) A particle, of mass m , placed in a conservative field generated by the potential function $U(x_1, x_2, x_3)$, $U \in C^1$.

The equations of motion are given by

$$(4) \quad m\ddot{x}_i = \frac{\partial U}{\partial x_i}$$

and we have the integral of energy

(5)

$$E - U = h,$$

where

$$E = \frac{1}{2} m (\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) = \frac{1}{2} m \dot{x}_i \dot{x}_i,$$

and h is a constant determined by the initial conditions.

ii) A conservative system, with n degrees of freedom, having a potential $U(q^1, q^2, \dots, q^n)$, $U \in C^1$, such that the motion of the system is described by the equations [4]

$$(6) \quad \frac{d}{dt} \left\{ \frac{\partial F}{\partial \dot{q}^h} \right\} - \frac{\partial F}{\partial q^h} = \frac{\partial U}{\partial q^h} \quad (h = \overline{1, n}),$$

where E is the kinetic energy of the system

$$E = \frac{1}{2} g_{ij} \dot{q}^i \dot{q}^j;$$

the coefficients g_{ij} and the potential function U depend on Lagrange coordinates q^h only. Even in this case the integral of energy holds and it has also the form (5).

2. In the following we will try to derive the equations of motions (4) and (6) from some special variational principles, which will be taken in such a way that the trajectories arise as geodesics of a Riemannian space, the metric of these spaces having some common characteristics with metrics used in the theory of relativity [1], [3].

The resulting models will be called relativistic models for the studied problems [2].

In order to construct a relativistic model for the equations (4) let us consider the metric

$$(7) \quad ds^2 = B(H - U)(mc^2 dt^2 - m dx_i dx_i),$$

where c is the speed of light, B and H are constants which will be determined later.

The geodesics of metric (7) are given by the following variational problem:

$$(8) \quad \delta \int \sqrt{B(H - U)(mc^2 - m \dot{x}_i \dot{x}_i)} dt = 0.$$

Integral (3) holds and we may write

$$\sqrt{B(H - U)(mc^2 - m \dot{x}_i \dot{x}_i)} + \frac{B(H - U) m \dot{x}_i \dot{x}_i}{\sqrt{B(H - U)(mc^2 - m \dot{x}_i \dot{x}_i)}} = A,$$

or

$$(9) \quad \sqrt{\frac{B(H-U)}{\Theta}} \quad mc^2 = a, \quad \Theta = mc^2 - m\dot{x}_i\dot{x}_i.$$

If we choose B such that

$$(10) \quad \frac{H-U}{\Theta} = \frac{1}{2}$$

and we take

$$(11) \quad H = \frac{1}{2}mc^2 - h,$$

the integral (9) becomes

$$\frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) - U = h$$

and it coincides with (5).

The Euler-Lagrange equations of (8) are

$$\frac{d}{dt} \left\{ \sqrt{\frac{H-U}{\Theta}} m\dot{x}_i \right\} = \frac{1}{2} \sqrt{\frac{\Theta}{H-u}} \frac{\partial U}{\partial x_i}$$

and taking into account (10), we can see, they coincide with (4).

For Eqs. (6) let us write the metric

$$(12) \quad ds^2 = B(H-U)(mc^2 - g_{ij}dq^i dq^j),$$

where c , B , H have the same meaning as in the preceding case; m is the mass of the system.

The geodesics of (12) are given by the variational problem

$$(13) \quad \delta \int \sqrt{B(H-U)(mc^2 - 2E)} \quad dt = 0.$$

Because the system is scleronomic we have integral (3) and we can write it as

$$\sqrt{B(H-U)(mc^2 - 2E)} + \frac{B(H-U)}{\sqrt{B(H-U)(mc^2 - 2E)}} \frac{\partial E}{\partial \dot{q}^h} \dot{q}^h = A,$$

hence

$$\sqrt{\frac{B(H-U)}{\Theta}} \quad mc^2 = A, \quad \Theta = mc^2 - 2E.$$

Taking B such that we have (10), we obtain

$$H - U = \frac{1}{2}mc^2 - F.$$

Choosing H as in (11), the above integral becomes

$$\frac{1}{2} g_{ij} \dot{q}^i \dot{q}^j - U = h,$$

i.e. the integral of energy.

The Euler-Lagrange equations, which derive from (13), are

$$\frac{d}{dt} \left\{ \sqrt{\frac{H-U}{\Theta}} \frac{\partial E}{\partial \dot{q}^h} \right\} - \sqrt{\frac{H-U}{\Theta}} \frac{\partial E}{\partial q^h} = \frac{1}{2} \sqrt{\frac{\Theta}{H-U}} \frac{\partial U}{\partial q^h},$$

which, taking into account (10), come to (6).

Therefore, the relativistic model for Eqs. (4) is given by (7) and the relativistic model for Eqs. (6) by (12).

Received August 28, 1994

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MODELE RELATIVISTE PENTRU SISTEME NEWTONIENE CONSERVATIVE

(Rezumat)

Se studiază unele modele conservative de mecanici newtoniene în vederea stabilirii conexiunilor cu teoria relativității.

SPHERICALLY SYMMETRIC SPACE-TIME IN THE CONTEXT OF THE U_4 THEORY

BY

SERVILIA OANCEA and V. MANTA

1. Introduction. The General Theory of Relativity based on Riemann-Cartan space-times U_4 is used for the study of the gravitational field of spherical symmetry. When the nestationary field is considered, the paper demonstrates that a separation of variables is not possible.

The finding of exact solution for the gravitational field in space-times with torsion is an open problem due to the fact that, at present, there is no unitary theory of gravitational field developed on Riemann-Cartan spaces. Both the classical theories as U_4 theory [1] and the gauge Poincaré theories of gravitation have been applied to obtain the exact solution. That is why this problem has interested us too, lately.

In the present paper we deal with the spherical symmetry because this field is much studied in literature, and in the same time it is easier to be approached. Both in case of the field equations of the U_4 theory that we used here and which have the form [2]

$$(1) \quad G^{ij} - \left(\nabla_k + K_{\alpha k}^{\alpha} \right) \left(K^{ijk} + g^{ij} K^{mk}_m - g^{ik} K^{mj}_m \right) = \chi T^{ij},$$

(where G^{ij} is the Einstein tensor, K^{ijk} is the contortion, ∇_k is the covariant derivative and T^{ij} — the energy-momentum tensor) and in the case of the gauge theory, the number of equations is smaller than the number of unknowns. If in Einstein's Theory of Relativity there are 10 fields equations and also 10 unknowns (the components of the metric tensor) in case of U_4 theory there are 16 equations (due to the fact that the Einstein tensor is not symmetric anymore) but there appear 24 contortion elements more. Thus in solving the field equations we start either from some physical aspects which determine some components of the torsion using the second set of equations of the form

$$(2) \quad K_{ijk} = -\chi \left(\tau_{ijk} - \tau_{jki} + \tau_{kij} + g_{ij} \tau_{\alpha k}^{\alpha} - g_{ik} \tau_{\alpha j}^{\alpha} \right),$$

or some constraints are imposed to torsion. A small number of components of the contortion can be chosen, as we have done up now [3]. We note that the great number of equations and the great working volume make the problem general approach impossible.

The selection of the number of non-null components which can characterize the spherically symmetry is also an important problem. Thus in [4], 6 elements are chosen for torsion, while in [5] only 4 elements are chosen.

In the following we aim at showing that in case of choosing 4 non-null components of the contortion tensor, for spherically symmetric field, a compatible system is obtained, where the number of the independent equations is equal to the number of unknowns.

2. Results. Let a spherically symmetric gravitational field be

$$(3) \quad ds^2 = e^{\lambda} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\varphi^2) - e^{-\lambda} dt^2,$$

with the coordinates $x^1 = r$, $x^2 = \theta$, $x^3 = \varphi$, $x^4 = t$ and where the function λ depend on r and t .

The following non-null components of the contortion are chosen:

$$(4) \quad \begin{aligned} K_{11}^1 &= -f, & K_{14}^1 &= -h, \\ K_{41}^4 &= -p, & K_{44}^4 &= -n, \end{aligned}$$

which are r and t functions.

We choose the elements with the indices 1 and 4 because these ones are related to the variables $x^1 = r$ and $x^4 = t$.

With components (4) of the contortion and with the general connection

$$(5) \quad \Gamma_{ij}^k = \left\{ \begin{matrix} k \\ i \quad j \end{matrix} \right\} + K_{ij}^k,$$

(where $\left\{ \begin{matrix} k \\ i \quad j \end{matrix} \right\}$ are Christoffel's symbols for (3)) the gravitational field equations (1) in the vacuum are (obtained using the algebraic computing system "REDUCE"):

$$(6a) \quad f'/-e^{2\lambda}\dot{h}-\frac{1}{2}e^{2\lambda}\dot{\lambda}h-e^{2\lambda}\dot{\lambda}n-\lambda'/f-\frac{1}{2}\lambda'/p-\frac{\lambda'}{r}-e^{2\lambda}\dot{n}+2e^{2\lambda}h^2+3e^{2\lambda}hn+ \\ +2e^{2\lambda}n^2-\frac{e^\lambda}{r^2}-4f^2-3fp-p^2+2\frac{f}{r}+\frac{2p}{r}+\frac{1}{r^2}=0,$$

$$(6b) \quad \dot{f}-h'/+\dot{\lambda}p+\frac{\dot{\lambda}}{r}-\lambda'h-n'/-fh+fn-2hp+np=0,$$

$$(6c) \quad \dot{f}-\dot{\lambda}p-\frac{\dot{\lambda}}{r}+\lambda'h-n'/+\dot{p}-fh-fn+2hp+np=0,$$

$$(6d) \quad f'/-e^{2\lambda}\dot{h}-\frac{1}{2}e^{2\lambda}\ddot{\lambda}-\frac{1}{2}e^{2\lambda}\dot{\lambda}^2-\frac{3}{2}e^{2\lambda}\dot{\lambda}h-e^{2\lambda}\dot{\lambda}n-\frac{1}{2}\lambda'/+\left(\frac{1}{2}\lambda'\right)^2-\lambda'/f- \\ -\frac{3}{2}\lambda'/p-\frac{\lambda'}{r}-e^{2\lambda}\dot{n}+p'/+2e^{2\lambda}h^2+4e^{2\lambda}hn+2e^{2\lambda}n^2-2f^2-4fp- \\ -2p^2+2\frac{f}{r}+\frac{p}{r}=0,$$

$$(6e) \quad f'/-\frac{1}{2}e^{2\lambda}\dot{\lambda}h-e^{2\lambda}\dot{\lambda}n-\lambda'/f-\frac{1}{2}\lambda'/p-\frac{\lambda'}{r}-e^{2\lambda}\dot{n}+p'/+e^{2\lambda}h^2+3e^{2\lambda}hn+ \\ +4e^{2\lambda}n^2-\frac{e^\lambda}{r^2}-2f^2-3pf-2p^2+2\frac{f}{r}+2\frac{p}{r}+\frac{1}{r^2}=0,$$

where $\dot{f} = \partial f / \partial t$, $f' = \partial f / \partial r$, etc.

From the equations (6a) and (6e), by subtraction, we obtain

$$(7) \quad -e^{2\lambda}\dot{h}-p'/+e^{2\lambda}h^2-2e^{2\lambda}n^2-2f^2+p^2=0.$$

In the same manner, from (6b) and (6c), by addition, we get

$$(8) \quad 2\dot{f}-h'-2n'/+\dot{p}-2fh+2np=0.$$

The Eqs. (7) and (8) can be used as starting points for selecting some particular cases.

The solving of the system (6) with 5 equations and 5 unknowns (f , p , h , n and λ) leads to the finding of the metric and of the contortion components.

It is noticed that the solving is extremely difficult because a separation of the variable r and t is not observed as in the case of a space with absolute parallelism. If in case of Riemann space such a separation is not noticed, the more in case of the space with curvature and torsion such a separation is expected to be impossible to obtain [6]. Even in case the static field is considered ($\dot{\lambda}=0$) and the functions f , p , h , n should not depend on time, the system of Eqs. (6) is still difficult to solve.

In order to simplify the system (6), we have chosen the case of the static field and consider 6 computing variants of the form:

$$(9a) \quad h=p=0, \quad f \neq 0, \quad n \neq 0;$$

$$(9b) \quad f=n=0, \quad h \neq 0, \quad p \neq 0;$$

$$(9c) \quad h=n=0, \quad f \neq 0, \quad p \neq 0;$$

$$(9d) \quad h=n=0, \quad f=p;$$

$$(9e) \quad f=p, \quad h=-n;$$

$$(9f) \quad f=p=n=h.$$

All these cases made the system (6) incompatible that leads to the idea that the selection of the particular cases modifies the field properties and the selection leads to properties which exclude each another. The 4 components of the contortion should be considered. The fact that some results from literature have been obtained, where such situations are also discussed, shows that it is no lack of interest to study spherically symmetric fields on Riemann-Cartan space-times.

It might be possible that the only admitted solution in the stationary case might be $f=p=h=n=0$ and then the solution is Schwarzschild metric, which would show that, for the spherical symmetry in vacuum, there is no other metric, even if one works in a more general space.

Received May 21, 1993

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SPAȚIU-TIMP CU SIMETRIE SFERICĂ ÎN CONTEXTUL TEORIEI U_4

(Rezumat)

În studiul câmpului gravitațional cu simetrie sferică se folosește Teoria Relativității Generale bazată pe un spațiu-timp Riemann-Cartan U_4 . Se arată că atunci când câmpurile considerate sunt nestaționare, separarea variabilelor nu este posibilă.

PLANE WAVES IN A NEW THEORY OF THERMOELASTICITY WITH MICROSTRUCTURE

BY

SILVIA GABRIELA CIUMAȘU

1. Introduction. In [3], a generalized theory of linear thermoelasticity with microstructure has been derived. In this paper we consider the problem of plane waves in the case of homogeneous and isotropic thermoelastic with microstructure solids. The study of plane waves in a new theory of thermoelasticity was given by E. B o s c h i [1], and in the linear elasticity with microstructure by M i n d l i n [4] and S o o s [5].

2. Basic Equations. Let be an homogeneous, isotropic, centrosymmetric, unlimited elastic solid with microstructure. We refer the motion of the solid to a fixed system of rectangular cartesian axes, Ox_i ($i=1, 2, 3$).

The basic equations in the absence of the body force, body double force and heat supply, are:

i) the equations of motion

$$(2.1) \quad t_{ji,j} = \rho \ddot{u}_i, \quad \sigma_{ij} + \mu_{kij,k} = \frac{\rho^*}{3} d^2 \ddot{\varphi}_{ij}, \quad t_{ij} = \tau_{ij} + \sigma_{ij};$$

ii) the equation of energy

$$(2.2) \quad \theta_0 \rho \dot{S} + q_{i,i} = 0;$$

iii) the constitutive equations

$$(2.3) \quad \begin{aligned} \tau_{ij} &= P \delta_{ij} (\theta + \alpha \dot{\theta}) + \lambda_1 \delta_{ij} \varepsilon_{rr} + 2 \mu_1 \varepsilon_{ij} + g_1 \delta_{ij} \gamma_{rr} + g_2 (\gamma_{ij} + \gamma_{ji}), \\ \sigma_{ji} &= R \delta_{ij} (\theta + \alpha \dot{\theta}) + g_1 \delta_{ji} \varepsilon_{rr} + 2 g_2 \varepsilon_{ji} + b_1 \delta_{ji} \gamma_{rr} + b_2 \gamma_{ji} + b_3 \gamma_{ij}, \\ \mu_{kij} &= a_1 (\varepsilon_{rrk} \delta_{ij} + \varepsilon_{jrr} \delta_{ki}) + a_2 (\varepsilon_{rri} \delta_{kj} + \varepsilon_{rjr} \delta_{ki}) + a_3 \varepsilon_{rrj} \delta_{ki} + \\ &\quad + a_4 \varepsilon_{krr} \delta_{ij} + a_5 (\varepsilon_{irr} \delta_{kj} + \varepsilon_{rkr} \delta_{ij}) + a_8 \varepsilon_{rir} \delta_{kj} + a_{10} \varepsilon_{kij} + \\ &\quad + a_{11} (\varepsilon_{jki} + \varepsilon_{ijk}) + a_{13} \varepsilon_{kji} + a_{14} \varepsilon_{ikj} + a_{15} \varepsilon_{jik}; \end{aligned}$$

$$q_i = -\theta_0 K \theta_{,i}, \quad \rho S = a + d\dot{\theta} + h\ddot{\theta} - P\epsilon_{ii} - R\gamma_{ii};$$

iv) the geometrical equations

$$(2.4) \quad \epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \gamma_{ij} = u_{j,i} - \varphi_{ij}, \quad \kappa_{ijk} = \varphi_{jk,i}.$$

We have used the notations from [3]. The constants P , R are given by the relations

$$P = \frac{1}{3}A(3\lambda + 2\mu) + B(3g_1 + 2g_2),$$

$$R = \frac{1}{3}[A(3g_1 + 2g_2) + B(3b_1 + b_2 + b_3)],$$

where A and B are coefficients of volume expansion at constant stress and micro-stress average. We assume that

$$(2.5) \quad 3\lambda_1 + 2\mu_1 > 0, \quad 3b_1 + b_2 + b_3 > 0,$$

$$3g_1 + 2g_2 > 0, \quad b_2 + b_3 > 0.$$

3. Plane Waves. We consider solutions of (2.1) – (2.4) of the form

$$(3.1) \quad u_i = u_i(x_1, t), \quad \varphi_{ij} = \varphi_{ij}(x_1, t), \quad \theta = \theta(x_1, t).$$

By means of linear combinations, the Eqs. (2.1), (2.2) may be separated into three independent equations and three independent systems. The temperature θ is coupled with the functions $u_1, \varphi = \frac{1}{3}\varphi_{ii}$, $\varphi_{ii}^D = \varphi_{ii} - \varphi$ in the system

$$(\theta_{,1} + \alpha\dot{\theta})(P + R) + k_{11}u_{1,11} - k_{12}\varphi_{11,11}^D - k_{13}\varphi_{,11} = \rho\ddot{u}_1,$$

$$(3.2) \quad 3R(\theta + \alpha\dot{\theta}) + k_{31}u_{1,1} + k_{32}\varphi_{11,11}^D + k_{33}\varphi_{,11} - k_{33}'\varphi = I\ddot{\varphi},$$

$$k_{21}u_{1,1} + k_{22}\varphi_{11,11}^D - k_{22}'\varphi_{11}^D + k_{23}\varphi_{,11} = \frac{1}{2}I\ddot{\varphi}_{11}^D,$$

$$k\theta_{,11} - d\dot{\theta} - h\ddot{\theta} + (P + R)\dot{u}_{1,1} - 3R\dot{\varphi} = 0, \quad (I = \varphi^* d^2),$$

where we have used the notations from [4], for the coefficients.

We seek the solutions of the form

$$(3.3) \quad u_s = A_s e^{i(\xi x_1 - \omega t)}, \quad \varphi_{sj} = B_{sj} e^{i(\xi x_1 - \omega t)}, \quad \theta = T_0 e^{i(\xi x_1 - \omega t)},$$

where A_s , B_{sj} , T_0 are complex constants and ξ is the wave number. We assume that ξ is a real constant.

Substitution of (3.3) into (3.2) yields the dispersion relation

$$\begin{vmatrix} (i\xi + \xi w\alpha)(P+R) & -k_{11}\xi^2 + \rho w^2 & -i\xi k_{12} & -i\xi k_{13} \\ 3R(1-i\alpha w) & i\xi k_{31} & -k_{32}\xi^2 & -k_{33}\xi^2 - k'_{33} + Iw^2 \\ 0 & i\xi k_{21} & -k_{22}\xi^2 - k'_{22} + \frac{I}{2}w^2 & -k_{23}\xi^2 \\ -k\xi^2 + idw + hw^2 & (P+R)\xi w & 0 & 3iRw \end{vmatrix} = 0.$$

If we consider the case of an imaginary medium for which $P=R=0$ then we obtain "uncoupled" waves: $u_s = u_s(x_1, t)$, $\varphi_{sj} = \varphi_{sj}(x_1, t)$ will be the purely elastic waves [4], and $\theta = \theta(x_1, t)$ the purely thermal wave.

For $P \neq 0$, $R \neq 0$ from (3.4) we obtain

$$\begin{aligned} (3.5) \quad & S_{21}w^8 + S_{20}w^7 + (S_{19}\xi^2 + S_{17})w^6 + (S_{18}\xi^2 + S_{16})w^5 + \\ & + (S_{15}\xi^4 + S_{13}\xi^2 + S_{11})w^4 + (S_{14}\xi^4 + S_{12}\xi^2 + S_{10})w^3 + \\ & + (S_9\xi^6 + S_7\xi^4 + S_5\xi^2)w^2 + (S_8\xi^6 + S_6\xi^4 + S_4\xi^2)w + \\ & + (S_1\xi^4 + S_2\xi^6 + S_3\xi^8) = 0, \end{aligned}$$

where S_i ($i=1, 2, \dots, 21$) depend on the material coefficients.

If we consider $\xi=0$ in (3.5) then we obtain

$$\begin{aligned} (3.6) \quad & \rho w^3(0) \left[-k'_{22} + \frac{1}{2}Iw^2(0) \right] \left[-hIw^3(0) - dIiw^2(0) + (9R^2\alpha + hk'_{33})\omega(0) + \right. \\ & \left. + (9R^2 + dk'_{33})i \right] = 0. \end{aligned}$$

From (3.6) we deduce

$$(3.7) \quad w_{1,2,3}(0) = 0, \quad w_{4,5}(0) = \pm \sqrt{\frac{2k'_{22}}{I}},$$

$$(3.8) \quad w^3(0) + \frac{d}{h}iw^2(0) - w(0) \left[\frac{9R^2\alpha}{hI} + \frac{k'_{33}}{I} \right] - i \left[\frac{9R^2}{hI} + \frac{dk'_{33}}{hI} \right] = 0.$$

For a cubic equation in w the roots may be written as

$$(3.9) \quad w_6(0) = f - ig, \quad w_7(0) = -f - ig, \quad w_8(0) = -il,$$

where g is the real root of the cubic equation

$$(3.10) \quad 8g^3 - 8g^2 \frac{d}{h} + 2g \left[\frac{d^2}{h^2} + \frac{9R^2\alpha}{hI} + \frac{k'_{33}}{I} \right] - \frac{9R^2}{h^2 I} (\alpha d - h) = 0,$$

with

$$(3.11) \quad l = \frac{d}{h} - 2g, \quad f^2 = 3g^2 - 2g \frac{d}{h} + \frac{9R^2\alpha}{hI} + \frac{k'_{33}}{I}.$$

In the case $P=R=0$, the only real root of (3.10) is $g=0$, and we have

$$(3.12) \quad w_{6,7}(0) = \pm \sqrt{\frac{k'_{33}}{I}}, \quad w_8(0) = -\frac{id}{h}.$$

The behaviour of the branches, at low frequencies, in the case of long wave length, can be described by

$$(3.13) \quad w(\xi) = w(0) + \xi w'(0) + \frac{\xi^2}{2!} w''(0) + \frac{\xi^3}{3!} w'''(0) + \dots$$

The values, at $w=0$ and $\xi=0$, of the derivatives of w with respect to ξ , are

$$(3.14) \quad w'_{1,2}(0) = \pm \left[\frac{E}{\rho k'_{22}(9R^2 + dk'_{33})} + \frac{dk'_{33}}{9R^2 + dk'_{33}} \left(w'_{LA}(0) \right)^2 \right]^{1/2},$$

$$(3.15) \quad w''_{1,2}(0) = -\frac{i}{U} [(\alpha d - h)L + kS],$$

$$(3.16) \quad w'''_{1,2}(0) = \frac{1}{S_4} \left\{ 3 \left[\left(w'_{1,2}(0) \right)^5 S_{16} + \left(w'_{1,2}(0) \right)^3 S_{12} + w'_{1,2}(0) S_6 \right] + \right.$$

$$\left. + w'_{1,2}(0) w''_{1,2}(0) \left[3S_5 + 2,25 w''_{1,2}(0) S_{10} + 6 \left(w'_{1,2}(0) \right)^3 S_{11} \right] \right\},$$

$$(3.17) \quad w'_3(0) = 0, \quad w''_3(0) = -\frac{2k}{S_4} \left(k_{11} k'_{22} k'_{33} - k_{31}^2 k'_{22} - k_{21}^2 k'_{33} \right),$$

$$(3.18) \quad w'''_3(0) = 0,$$

$$(3.18) \quad w^{IV}_3(0) = \frac{-3}{S_4} \left[8S_2 + 4S_6 w''_3(0) + 2 \left(w''_3(0) \right)^2 S_5 + 3 \left(w''_3(0) \right)^3 S_{10} \right],$$

where

$$E = P^2 k'_{22} k'_{22} + 2 P k'_{22} (k'_{33} - 3 k_{31}) + R^2 (k'_{22} k'_{33} - 6 k_{31} k'_{22} + 9 k_{11} k'_{22} - 9 k_{12}^2),$$

$$\left(w_{LA}^{(e)}(0) \right)^2 = \frac{1}{\rho k'_{22} k'_{33}} \left(k_{11} k'_{22} k'_{33} - k_{31}^2 k'_{22} - k_{21}^2 k'_{33} \right) = F,$$

$$U = \rho k'_{22} (9 R^2 + d k'_{33}) (E + d F A),$$

$$S = (9 R^2 \rho k'_{22} + d \rho A) \left[(E + d F A) \rho A - F A (9 R^2 \rho k'_{22} + d \rho A) \right],$$

$$L = (P + R)^2 A \rho \left[(P + R)^2 (A^2 + 6 R A B) + 36 R^2 B^2 + \alpha A^2 F - 18 R^2 A C \right] - \\ - 9 R^2 \rho \left[k'_{22} F A (\alpha F A + 9 A + 9 R^2 C) + 81 R (P + R) B C A \right],$$

$$A = k'_{22} k'_{33}, \quad B = k_{31} k'_{22}, \quad C = k_{11} k'_{22} - k_{12}^2,$$

$$S_2 = k \left(k_{11} k'_{22} k'_{33} - k_{11} k'_{33} k'_{22} + 2 k_{31} k_{12} k_{23} - k_{31}^2 k'_{22} - k_{21}^2 k'_{33} \right),$$

$$S_4 = -i \left[(P + R)^2 A - 6 R (P + R) B + 9 R^2 C + \frac{d}{\rho} F A \right],$$

$$S_5 = 6 R (P + R) \alpha B - (P + R)^2 \alpha A - 9 R^2 \alpha C - k \rho A - \frac{h}{\rho} F A,$$

$$S_6 = i \left[6 R (P + R) (k_{31} k_{12} - k_{21} k_{32}) - (P + R)^2 (k'_{22} k'_{33} + k_{22} k'_{33}) - 9 R^2 k_{11} k'_{22} - \right. \\ \left. - d (k_{11} k'_{22} k'_{33} + k_{11} k_{22} k'_{33} + k_{31} k_{12} k_{23} - k_{31}^2 k'_{22} - k_{21}^2 k'_{33}) \right],$$

$$S_{10} = i \rho (9 R^2 + d k'_{33}) k'_{22}, \quad S_{11} = \rho k'_{22} (9 R^2 \alpha + h k'_{33}),$$

$$S_{12} = i \left[(P + R)^2 \frac{1}{2} I (k'_{33} + 2 k'_{22}) + 9 R^2 \left(\frac{1}{2} I k_{11} + \rho k_{22} \right) - 3 R (P + R) I k_{31} + \right. \\ \left. + \frac{1}{2} d I (k_{11} k'_{13} + 2 k_{11} k'_{22} - k_{31}^2 - 2 k_{12}^2) + d \rho (k'_{22} k'_{33} + k_{22} k'_{33}) \right],$$

$$S_{16} = -i \frac{1}{2} I \rho \left[9 R^2 + d (k'_{33} + k'_{22}) \right].$$

With $w^{(e)}$ we denote a value corresponding to purely elastic waves.

If are satisfied the conditions of uniqueness theorem [2] and conditions (2.5), then $w_{1,2}'(0)$ are real constants. In the case of purely elastic waves, $w_{1,2}^{(e)''}(0)=0$, (so that $\alpha=k=h=0$). If $[(\alpha d-h)L+kS]/U > 0$, $w_{1,2}^{(e)''}(0)$ represents an attenuation coefficient in time of the wave. The assumption that $[(\alpha d-h)L+kS]/U < 0$ is physically unacceptable.

From (3.16) it follows that $w_{1,2}^{(e)''''}(0)$ is a real constant, and from (3.17), (3.18) $w_3^{(e)''}(0)$ and $w_3^{(e)''''}(0)$ are imaginary.

Examining these results, for high frequencies, we see that the root $w_{1,2}$ corresponds to a modification of the purely acoustic elastic longitudinal branche. The low frequencies for the longitudinal acoustic branche are given by

$$\Re w_{LA}(\xi) = w_{1,2}'(0)\xi + w_{1,2}^{(e)''}(0)\frac{\xi^3}{3!}$$

and the attenuation in time by respectively

$$w_{LA}(\xi) = w_{1,2}^{(e)''}(0)\frac{\xi^2}{2}.$$

The group velocity of the acoustic elastic modified branche is

$$(3.19) \quad V_L(\xi) = V_L \left\{ 1 + \frac{2\pi^2}{\lambda^2} \left[\frac{1}{S_4} \left[3(w_{1,2}'(0))^4 S_{16} + (w_{1,2}'(0))^2 S_{12} + S_6 \right] + \right. \right. \\ \left. \left. + w_{1,2}^{(e)''}(0) \left[3S_5 + 2.25 w_{1,2}^{(e)''}(0) S_{10} + 6(w_{1,2}'(0))^2 S_{11} \right] \right] \right\},$$

where

$$(3.20) \quad V_L^2 = V_L^2(0) = \frac{E}{\rho k_{22}'(9R^2 + dk_{33}')} + V_L^{2(e)}(0) \frac{dk_{33}'}{9R^2 + dk_{33}'}.$$

The root $\omega_3(\xi)$ corresponds to a modified purely acoustic thermal wave. We see that the amplitude decays exponentially with time. It is proportional to $e^{-w_3(\xi)t}$, where

$$w_3(\xi) = w_3^{(e)''}(0)\frac{\xi^2}{2} + w_3^{(e)''''}(0)\frac{\xi^4}{4!}.$$

For optical branches $w_0(\xi)$, which in $\xi=0$ have the values (3.7)₂, (3.9); the values at $w \neq 0$, $\xi=0$, of the derivatives of w with respect to ξ , are

$$(3.21) \quad (w'(0))^2 = \frac{E_1}{E_2}, \quad w_0''(0) = \frac{E_3(\omega_0'(0))^2 + E_4}{E_2 \omega_0(0)},$$

where

$$E_1 = S_{19} w_0^5(0) + S_{18} w_0^4(0) + S_{13} w_0^3(0) + S_{12} w_0^2(0) + S_5 w_0(0) + S_4,$$

$$E_2 = -\left(28 S_{21} w_0^5(0) + 21 S_{20} w_0^4(0) + 15 S_{17} w_0^3(0) + 10 S_{16} w_0^2(0) + \right. \\ \left. + 3 w_0(0) S_{11} + 3 S_{10}\right),$$

$$E_3 = 56 S_{21} w_0^5(0) + 35 S_{20} w_0^4(0) + 40 S_{17} w_0^3(0) + 10 S_{16} w_0^2(0) + \\ + 4 w_0(0) S_{11} + S_{10},$$

$$E_4 = 6 w_0^5(0) S_{19} + 5 w_0^4(0) S_{18} + 3 w_0^3(0) S_{13} + 3 w_0^2(0) S_{12} + 2 w_0(0) S_5 + S_4,$$

$$S_{13} = (P+R)^2 \alpha \frac{1}{2} I (k_{33}' + 2k_{22}') + 9R^2 \alpha \left(\frac{1}{2} I k_{11} + \rho k_{22} \right) - \\ - 3R(P+R) I \alpha k_{31} + \frac{1}{2} k \rho I (k_{33}' + 2k_{22}') + \frac{1}{2} V_2 I (k_{11} k_{33}' + 2k_{11} k_{22}' - \\ - k_{31}^2 - 2k_{12}^2) + h \rho (k_{22}' k_{33} + k_{22} k_{33}'),$$

$$S_{17} = -\frac{1}{2} \rho I \left[9R^2 \alpha + h(k_{33}' + k_{22}') \right],$$

$$S_{18} = -i \frac{1}{2} I \left[(P+R)^2 I + d k_{11} I + d \rho (k_{33} + k_{22}) \right],$$

$$S_{19} = -\frac{1}{2} I \left[(P+R)^2 \alpha I + k \rho I + h k_{11} I + h \rho (k_{33} + 2k_{22}) \right],$$

$$S_{20} = \frac{1}{2} i d \rho I^2, \quad S_{21} = \frac{1}{2} h \rho I^2.$$

For (3.7)₂, (3.9), from (3.17) we obtain complex constants. The frequencies for optical longitudinal branches are given by: $w_0(\xi) = \Re w_0(0) + V_0(\xi)\xi$, where $V_0(\xi)$ is the group velocity given by $V_0(\xi) = \Re w_0'(0) + \xi \Re w_0''(0)$.

4. Conclusions. We find that there are three optical branches:

i) a longitudinal optical, shear, elastic modified wave, which has frequency in $\xi=0$ equal with $w_{4,5}(0) = \pm \sqrt{2k_{22}/I}$;

ii) a longitudinal optical dilatational elastic modified wave, which has frequency in $\xi=0$, $w_{6,7}(0) = \pm f$;

iii) a longitudinal optical, thermal modified wave; this motion exists as a progressive wave, with the amplitude which decays exponentially in time, only if $w_0(0)$ is a real constant.

Both the acoustic and the optical thermoelastic branches are dispersed and attenuated in time.

Received March 7, 1991

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UNDE PLANE ÎN NOUA TEORIE A TERMOELASTICITĂȚII CU MICROSTRUCTURĂ

(Rezumat)

Pentru un solid nemărginit, omogen, izotrop, cu centru de simetrie în fiecare punct, se face un studiu calitativ asupra propagării undelor plane, în cadrul teoriei termoelasticității cu microstructură, stabilită în [3]. Se determină relația de dispersie pentru undele longitudinale și se pun în evidență fenomenele de dispersie și de amortizare a acestora.

LINEAR THERMOELASTICITY IN THE THEORY
OF THE COSSERAT SURFACE

BY

GH. CHIORESCU

1. Introduction. By Cosserat surface we mean a surface embedded in Euclidian 3-space, to every point of which it is assigned a vector called director. A Cosserat surface is a mathematical model for a shell. The director vector represents the variation of the transversal section.

The general theory of the Cosserat surfaces was developed in the papers [1], [2], [3]. In the papers [1] and [2] are considered three director vectors. We refer to the general theory as it was treated in the paper [3]. In that paper are presented the kinematical results for Cosserat surface and by using the conservation laws, the basic equations have been deduced.

The elastic linear theory for some particular cases (isotropic material, particular deformation) was treated in the paper [3]. In the following, we present a linear theory by considering the temperature, without other restrictive assumption (above mentioned). Then for the theory presented we shall define the weak solution and we shall demonstrate a Friedrichs like inequality.

2. Basic. Equations. Let φ be a surface defined by a uniform differentiable application from an open bounded set $\Omega \subset \mathbb{R}^2$ into a Euclidian 3-space E_3 . We suppose the boundary $\partial\varphi$ having tangent vector. A surface φ to every point of which (given by position vector $\mathbf{R}$) is assigned a material vector $\mathbf{D}$ (called director) is named Cosserat surface. The vector $\mathbf{D}$ deforms independently in respect to the points of the surface. The couple $(\varphi, \mathbf{D})$ is a mathematical model for a shell. The deformation of the surface gives information about the deformation of the median surface of the shell. The variation in magnitude or direction of the vector $\mathbf{D}$ gives information about the variation of the thickness or the orientation of the shell.

We suppose the vector $\mathbf{D}$ to be colinear to the normal of the surface φ and let θ^1, θ^2 be the surface parameters. Thus we have

$$(2.1) \quad \mathbf{D} = D(\theta^1, \theta^2) \mathbf{A}_3; \quad \mathbf{A}_3 = \frac{\mathbf{A}_1 \times \mathbf{A}_2}{|\mathbf{A}_1 \times \mathbf{A}_2|},$$

$$\mathbf{A}_1 = \frac{\partial \mathbf{R}}{\partial \theta^1}, \quad \mathbf{A}_2 = \frac{\partial \mathbf{R}}{\partial \theta^2}.$$

A shell with uniform thickness in the reference configuration is characterized by the condition $D(\theta^1, \theta^2) = \text{constant}$.

If $\mathbf{r}$, $\mathbf{d}$, T are the position vector, the director vector and the temperature in the deformed configuration and $\mathbf{R}$, $\mathbf{D}$, θ_0 are the same quantities in the reference configuration, then the *displacements* $\mathbf{u}$, δ , θ are defined by

$$(2.2) \quad \mathbf{r}(\theta^1, \theta^2) = \mathbf{R}(\theta^1, \theta^2) + \mathbf{u}(\theta^1, \theta^2),$$

$$\mathbf{d}(\theta^1, \theta^2) = \mathbf{D}(\theta^1, \theta^2) + \delta(\theta^1, \theta^2),$$

$$T(\theta^1, \theta^2) = \theta_0(\theta^1, \theta^2) + \theta(\theta^1, \theta^2).$$

The *deformation measures* in the linear elastic case are given by

$$(2.3) \quad e_{\alpha\beta} = \frac{1}{2}(u_{\alpha/\gamma} + u_{\gamma/\alpha}) - B_{\alpha\gamma} u_3,$$

$$\kappa_{\gamma\alpha} = D_{,\alpha} B_{\gamma}^{\nu} \delta_{\nu} + D B_{\alpha}^{\beta} B_{\gamma\beta} u_3 - B_{\alpha\gamma} \delta_3 - D B_{\alpha}^{\beta} u_{\beta\gamma} + D_{,\alpha} u_{3,\gamma} + \delta_{\gamma/\alpha},$$

$$s_{\alpha} = D^2 B_{\alpha}^{\nu} B_{\nu}^{\delta} u_{\delta} + D B_{\alpha}^{\nu} \delta_{\nu} + D^2 B_{\alpha}^{\nu} u_{3,\nu} + D \delta_{3,\alpha} + D_{,\alpha} \delta_3,$$

$$\gamma_{\alpha} = D B_{\alpha}^{\nu} u_{\nu} + \delta_{\alpha} + D u_{3,\alpha},$$

$$s = 2D \delta_3,$$

where a comma denotes the partial derivative, a bar stays for the covariant derivation with respect to the metric tensor $A_{\alpha\beta} = \mathbf{A}_{\alpha} \cdot \mathbf{A}_{\beta}$ and $B_{\alpha\beta}$ are the coefficient of the second fundamental form of the surface φ . Also, we include as a deformation measure, the gradient of the temperature $\theta_{,\beta}$.

We consider a *thermoelastic material* defined by the constitutive equations

$$(2.4) \quad N^{\alpha\beta} + D M^{\gamma\alpha} B_{\gamma}^{\beta} = \rho_0 \frac{\partial \psi}{\partial e_{\alpha\beta}},$$

$$m^{\alpha} = \rho_0 \frac{\partial \psi}{\partial \gamma_{\alpha}}, \quad m^3 = \rho_0 \left[2D \frac{\partial \psi}{\partial s} + D_{,\alpha} \frac{\partial \psi}{\partial s_{\alpha}} \right],$$

$$M^{\alpha\beta} = \rho_0 \frac{\partial \psi}{\partial \kappa_{\alpha\beta}}, \quad M^{\alpha 3} = \rho_0 D \frac{\partial \psi}{\partial s_{\alpha}},$$

$$N^{\alpha 3} - m^{\alpha} D - M^{\gamma 3} D B_{\gamma}^{\alpha} - M^{\gamma\alpha} D_{,\gamma} = 0,$$

$Q^\alpha = -k^{\alpha\beta}\theta_{,\beta}$, $k^{\alpha\beta} = k^{\beta\alpha}$, $k^{11} \geq 0$, $k^{22} \geq 0$, $k^{11}k^{22} - k^{12}k^{21} \geq 0$, where ψ is the Helmholtz free energy function; φ — the mass density in the reference configuration; $N^{\alpha\beta}$, $N^{\alpha 3}$ are the components of the contact force vectors $\mathbf{N}^\alpha$, measured per unit length in the reference configuration; $M^{\alpha\beta}$, $M^{\alpha 3}$ — the components of the director couples $\mathbf{M}^\alpha$, measured per unit length in the reference configuration; m^α , m^3 — the components of the director couple intrinsic $\mathbf{m}$, measured per unit area in the reference configuration; Q^α — the components of the flux vector per unit length in the reference configuration.

The components of the vectors above mentioned are considered in respect to the basis vectors $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$.

The equilibrium equations and the reduced energy equation are given by

$$(2.5) \quad \begin{aligned} N_{/\alpha}^{\alpha\beta} - B_\alpha^\beta N^{\alpha 3} + \rho_0 F^\beta &= 0, & N_{/\alpha}^{\alpha 3} + B_{\alpha\beta} N^{\alpha\beta} + \rho_0 F^3 &= 0, \\ M_{/\alpha}^{\alpha\beta} - B_\alpha^\beta M^{\alpha 3} + \rho_0 L^\beta &= m^\beta, & M_{/\alpha}^{\alpha 3} + B_{\alpha\beta} M^{\alpha\beta} + \rho_0 L^3 &= m^3, \\ Q_{/\alpha}^\alpha - \rho_0 r &= 0, \end{aligned}$$

where F^i , L^i are the components of the body force vector and the components of the couple mass vector, respectively, measured per unit area in the reference configuration and r in the source of heat intensity measured per unit area in the reference configuration.

Let the function $\varphi_0 \psi$ be given by the following quadratic form:

$$(2.6) \quad \begin{aligned} \rho_0 \psi &= A^{\alpha\beta\lambda\delta} e_{\alpha\beta} e_{\lambda\delta} + B^{\alpha\beta\lambda\delta} \kappa_{\alpha\beta} \kappa_{\lambda\delta} + C^{\alpha\beta\lambda\delta} e_{\alpha\beta} \kappa_{\lambda\delta} + \\ &+ D^{\alpha\beta\lambda} s_\alpha \kappa_{\beta\lambda} + E^{\alpha\beta\lambda} e_{\alpha\beta} \gamma_\lambda + F^{\alpha\beta\lambda} e_{\alpha\beta} s_\lambda + G^{\alpha\beta\lambda} \gamma_\alpha \kappa_{\beta\lambda} + \\ &+ H^{\alpha\beta} \gamma_\alpha \gamma_\beta + I^{\alpha\beta} s_\alpha s_\beta + J^{\alpha\beta} \gamma_\alpha s_\beta + K^{\alpha\beta} e_{\alpha\beta} s + L^{\alpha\beta} \kappa_{\alpha\beta} s + \\ &+ M^\alpha \gamma_\alpha s + N^\alpha s_\alpha s + O s^2 + P \theta^2, \end{aligned}$$

where the coefficients $A, \dots, P$ are functions which depend on θ^1 , θ^2 (anisotropic shell) and satisfy the following symmetries conditions:

$$(2.7) \quad \begin{aligned} A^{\alpha\beta\lambda\delta} &= A^{\beta\alpha\lambda\delta} = A^{\alpha\beta\delta\lambda} = A^{\lambda\delta\alpha\beta}, \\ B^{\alpha\beta\lambda\delta} &= B^{\lambda\delta\alpha\beta}, \quad C^{\alpha\beta\lambda\delta} = C^{\beta\alpha\lambda\delta}, \\ E^{\alpha\beta\lambda} &= E^{\beta\alpha\lambda}, \quad F^{\alpha\beta\lambda} = F^{\beta\alpha\lambda}, \quad G^{\alpha\beta\lambda} = G^{\beta\alpha\lambda}, \\ H^{\alpha\beta} &= H^{\beta\alpha}, \quad I^{\alpha\beta} = I^{\beta\alpha}, \quad J^{\alpha\beta} = J^{\beta\alpha}, \quad K^{\alpha\beta} = K^{\beta\alpha}. \end{aligned}$$

The absence in the energy function of the terms linear depending on the temperature is a consequence of the existence of a thermic symmetry center.

By introducing the function $\varphi_0 \psi$ given by (2.6) in the Eqs. (2.4), (2.5)

we obtain a system of 7 equations for 7 unknowns $\mathbf{h}=(u_1, u_2, u_3, \delta_1, \delta_2, \delta_3, \theta)$.

3. The Weak Formulation of a Boundary Value Problem. Let $\mathbf{h}=(u_1, u_2, u_3, \delta_1, \delta_2, \delta_3, \theta)$ be an arbitrary displacement vector, $\mathbf{h} \in C^1(\Omega)$ and let us denote by $\mathbf{f}$ the vector

$$(3.1) \quad \mathbf{f} = \rho_0(F^1, F^2, F^3, L^1, L^2, L^3, r).$$

Theorem 3.1. If a vector $\mathbf{h}$ is a solution of the equations (2.5) then the following relation holds

$$(3.2) \quad \int_{\varphi} \mathbf{f} \cdot \mathbf{h} / d\Sigma = \int_{\varphi} \left\{ \rho_0 \left[\frac{\partial \psi}{\partial e_{\alpha\beta}} e'_{\alpha\beta} + \frac{\partial \psi}{\partial \kappa_{\alpha\beta}} \kappa'_{\alpha\beta} + \frac{\partial \psi}{\partial s_{\alpha}} s'_{\alpha} + \frac{\partial \psi}{\partial \gamma_{\alpha}} \gamma'_{\alpha} + \frac{\partial \psi}{\partial s} s' \right] + \right. \\ \left. + k^{\alpha\beta} \theta_{,\beta} \theta'_{,\alpha} \right\} d\Sigma - \int_{\partial\varphi} \left[\rho_0 \left(\frac{\partial \psi}{\partial e_{\alpha\beta}} - \frac{\partial \psi}{\partial \kappa_{\alpha\lambda}} D_{\lambda}^{\beta} \right) u'_{\beta} + \right. \\ \left. + \rho_0 \frac{\partial \psi}{\partial \kappa_{\alpha\beta}} \delta'_{\beta} + \rho_0 \left(D_{,\beta} \frac{\partial \psi}{\partial \kappa_{\alpha\beta}} + D^2 B_{\beta}^{\alpha} \frac{\partial \psi}{\partial s_{\beta}} + D \frac{\partial \psi}{\partial \gamma_{\alpha}} \right) u'_{\beta} + \right. \\ \left. + \rho_0 \frac{\partial \psi}{\partial s_{\alpha}} \delta'_{\alpha} + k^{\alpha\beta} \theta_{,\beta} \theta'_{,\alpha} \right] \nu_{\alpha} dS,$$

where ν_{α} , $\alpha=1, 2$ are the parameters of the normal to the curve $\partial\varphi$ and dS is the arc element.

P r o o f. By substituting $\varphi_0 F^i$, $\varphi_0 L^i$, $\varphi_0 r$ from (2.5) and by using the constitutive equation we obtain

$$\int_{\varphi} \mathbf{f} \cdot \mathbf{h} / d\Sigma = \int_{\varphi} \left[\rho_0 \frac{\partial \psi}{\partial e_{\alpha\beta}} \left(\frac{1}{2} (u'_{\alpha/\beta} + u'_{\beta/\alpha}) - B_{\alpha\beta} u'_3 \right) + \rho_0 \frac{\partial \psi}{\partial \kappa_{\alpha\beta}} \left(-D B_{\beta}^{\lambda} u'_{\lambda/\alpha} + \right. \right. \\ \left. \left. + B_{\alpha}^{\lambda} D_{,\beta} u'_{\lambda} + D_{,\beta} u'_{3,\alpha} + D B_{\alpha\lambda} B_{\beta}^{\lambda} u'_3 + \delta'_{\alpha/\beta} - B_{\beta\alpha} \delta'_3 \right) + \right. \\ \left. + \rho_0 \frac{\partial \psi}{\partial s_{\alpha}} \left(D^2 B_{\alpha}^{\beta} u'_{3,\beta} + D^2 B_{\alpha}^{\beta} B_{\beta}^{\lambda} \mu'_{\lambda} + B_{\alpha}^{\beta} D \delta'_{\beta} + D \delta'_{3,\alpha} + D_{,\alpha} \delta'_3 \right) + \right. \\ \left. + \rho_0 \frac{\partial \psi}{\partial \gamma_{\alpha}} \left(D B_{\alpha}^{\beta} u'_{\beta} + D u'_{3,\alpha} + \delta'_{\alpha} \right) + \rho_0 \frac{\partial \psi}{\partial S} (2 D \delta'_3) + k^{\alpha\beta} \theta_{,\beta} \theta'_{,\alpha} \right] d\Sigma -$$

$$\begin{aligned}
& - \int_{\varphi} \left[\left(\rho_0 \frac{\partial \psi}{\partial e_{\alpha\beta}} u'_{\beta} \right)_{/\alpha} - \left(\rho_0 \frac{\partial \psi}{\partial x_{\alpha\beta}} D B_{\beta}^{\lambda} u'_{\lambda} \right)_{/\alpha} + \right. \\
& + \left(\rho_0 D_{,\beta} \frac{\partial \psi}{\partial x_{\alpha\beta}} u'_{\beta} \right)_{/\alpha} + \left(\rho_0 \frac{\partial \psi}{\partial x_{\alpha\beta}} \delta'_{\alpha} \right)_{/\beta} + \left(\rho_0 D^2 B_{\alpha}^{\beta} \frac{\partial \psi}{\partial s_{\alpha}} u'_{\beta} \right)_{/\beta} \\
& \left. + \left(\rho_0 D \frac{\partial \psi}{\partial s_{\alpha}} \delta'_{\beta} \right)_{/\alpha} + \left(\rho_0 D \frac{\partial \psi}{\partial \gamma_{\alpha}} u'_{\beta} \right)_{/\alpha} + (k^{\alpha\beta} \theta_{,\beta} \theta'_{,\alpha})_{/\alpha} \right] d\Sigma.
\end{aligned}$$

By applying the Stokes formula to the second surface integral in the right side member, the Eq. (3.2) is proved.

Let $\mathcal{L}_2(\varphi)$ be the closure of the function from $C^{\infty}(\bar{\varphi})$ in respect to the norm

$$(3.3) \quad \|h\|_0^2 = \int_{\varphi} \left(\sum_{i=1}^7 h_i^2 \right) d\Sigma, \quad d\Sigma = \sqrt{A} d\theta^1 d\theta^2, \quad A = \det \|A_{\alpha\beta}\|.$$

We suppose $f \in \mathcal{L}_2(\varphi)$.

Let $C_0^{\infty}(\varphi)$ be the space of the functions with compact support in φ so that if $h' \in C_0^{\infty}(\varphi)$ then $h' = 0$ on $\partial\varphi$. We consider the following bilinear form

$$\begin{aligned}
A(h, h') = \int_{\varphi} \left\{ \rho_0 \left[\frac{\partial \psi}{\partial e_{\alpha\beta}} e'_{\alpha\beta} + \frac{\partial \psi}{\partial x_{\alpha\beta}} x'_{\alpha\beta} + \frac{\partial \psi}{\partial s_{\alpha}} s'_{\alpha} + \right. \right. \\
\left. \left. + \frac{\partial \psi}{\partial \gamma_{\alpha}} \gamma'_{\alpha} + \frac{\partial \psi}{\partial s} s' \right] + k^{\alpha\beta} \theta_{,\beta} \theta'_{,\alpha} \right\} d\Sigma,
\end{aligned}$$

where $\varphi_0\psi$ is given by (2.6). The Eq. (3.2) may be written as

$$(3.4) \quad A(h, h') = \int_{\varphi} f \cdot h' d\Sigma, \quad \forall h \in C_0^{\infty}(\bar{\varphi}).$$

Let V be a vectorial space which will be defined, corresponding to the boundary value problem considered.

Definition. We call weak solution for a boundary value problem, a function $h \in V$ such that (3.4) holds.

Theorem 3.2. *If the thickness of the shell has a continuous variation ($D = D(\theta^1, \theta^2) \in C^1(\Omega)$) and the surface has continuous curvatures ($B_{\alpha\beta} \in C^1(\Omega)$) then the following Friedrichs like inequality holds*

$$(3.5) \quad \int_{\varphi} \left[\sum_{\alpha, \beta=1}^2 (e_{\alpha\beta}^2 + \kappa_{\alpha\beta}^2) + \sum_{\alpha=1}^2 (\gamma_{\alpha}^2 + s_{\alpha}^2 + \theta_{,\alpha}^2) + s^2 + \theta^2 \right] d\Sigma \geq k \|h\|_1^2,$$

$k = \text{const.},$

where

$$(3.6) \quad \|h\|_1^2 = \int_{\varphi} \left[\sum_{i=1}^7 h_i \right]^2 d\Sigma + \int_{\varphi} \left[\sum_{i, \alpha=1}^{7,2} h_{i, \alpha}^2 \right] d\Sigma.$$

P r o o f. By using the method from [4], in the paper [5] (Lemma 13.2) was been proved that in the above conditions (except the case of the membrane theory) the following inequality holds

$$\begin{aligned} \int_{\varphi} \left[\sum_{\alpha, \beta=1}^6 (e_{\alpha\beta}^2 + \kappa_{\alpha\beta}^2) + \sum_{\alpha=1}^2 (\gamma_{\alpha}^2 + s_{\alpha}^2) + a^2 \right] d\Sigma \geq \\ \geq k_1 \int_{\varphi} \left[\sum_{i=1}^6 h_i^2 + \sum_{i, \alpha=1}^{6,2} h_{i, \alpha}^2 \right] d\Sigma, \quad k_1 = \text{const.} \end{aligned}$$

By adding in the both members

$$\int_{\varphi} \left[\sum_{\alpha=1}^2 \theta_{,\alpha}^2 + \theta^2 \right] d\Sigma,$$

it results the inequality (3.6), where $k = \min(k_1, 1)$.

Received April 23, 1991

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TERMOELASTICITATE LINIARĂ ÎN TEORIA SUPRAFEȚELOR COSSERAT

(Rezumat)

O suprafață Cosserat este o suprafață care în fiecare punct al ei are atașată un vector deformabil numit director. Se presupune că vectorul director este situat de-a lungul normalei la configurația de referință. Pentru un astfel de corp se scriu ecuațiile de bază, se propune o funcție energie de deformare în cazul unui material anizotrop, se definește soluția slabă a unei probleme la limită în cadrul acestei teorii și se demonstrează o inegalitate de tip Friedrichs.

LA DÉFORMATION DANS LA THÉORIE DE LA
VISCOÉLASTICITÉ LINÉAIRE MICROPOLAIRE

PAR

D. MANOLE

1. Introduction. Dans la théorie de la viscoélasticité linéaire dans une dimension, la déformation optimale est établie dans [1]. Une propriété de monotonie dans cette théorie on trouve dans [2].

Dans ce qui suit on élargit les résultats de [1] dans une seule dimension.

2. Notions de base. Les équations constitutives pour un milieu viscoélastique micropolaire dans une dimension sont [3]

$$\sigma(t) = \int_0^t A(t-\tau) \dot{\gamma}(\tau) d\tau + \int_0^t B(t-\tau) \dot{\kappa}(\tau) d\tau, \quad (1.1)$$

$$m(t) = \int_0^t B(t-\tau) \dot{\gamma}(\tau) d\tau + \int_0^t C(t-\tau) \dot{\kappa}(\tau) d\tau,$$

où σ est la tension, m — couple de tension; γ, κ — composantes de la déformation; A, B, C — fonctions de relaxation; le point représente la dérivée par rapport au temps.

On va introduire les notations

$$(1.2) \quad p^T = [\gamma \quad \kappa], \quad F = \begin{bmatrix} A & B \\ B & C \end{bmatrix},$$

$$(1.3) \quad W(p) = \int_0^T \int_0^t \dot{p}^T(t) F(t-\tau) \dot{p}(\tau) d\tau dt,$$

où $0 \leq t \leq T$, T — fixé.

Si $f \in C[0, T]$, alors

$$\begin{aligned}
 (1.4) \quad & \int_0^T f^T(t) 2\dot{F}(0) f(t) dt + \int_0^T \int_0^T f^T(t) \ddot{F}(|t-\tau|) f(\tau) d\tau dt = \\
 & = -\frac{1}{2} \int_0^T \int_0^T (f(t) - f(\tau))^T \ddot{F}(|t-\tau|) (f(t) - f(\tau)) d\tau dt + \\
 & + \int_0^T f^T(t) (\dot{F}(t) + \dot{F}(T-t)) f(t) dt.
 \end{aligned}$$

En effet

$$\int_0^T \ddot{F}(|t-\tau|) dt = \int_0^\tau \ddot{F}(\tau-t) dt + \int_\tau^T \ddot{F}(t-\tau) dt = -2\dot{F}(0) + \dot{F}(\tau) + \dot{F}(T-\tau),$$

d'où résulte

$$\begin{aligned}
 (1.5) \quad & \int_0^T \int_0^T f^T(\tau) \ddot{F}(|t-\tau|) f(\tau) d\tau dt = - \int_0^T f^T(t) 2\dot{F}(0) f(t) dt + \\
 & + \int_0^T f^T(t) (\dot{F}(t) + \dot{F}(T-t)) f(t) dt;
 \end{aligned}$$

mais

$$\begin{aligned}
 (1.6) \quad & -\frac{1}{2} \int_0^T \int_0^T (f^T(t) - f^T(\tau)) \ddot{F}(|t-\tau|) (f(t) - f(\tau)) d\tau dt = \\
 & = \int_0^T \int_0^T f^T(t) \ddot{F}(|t-\tau|) f(\tau) d\tau dt - \int_0^T \int_0^T f^T(\tau) \ddot{F}(|t-\tau|) f(\tau) d\tau dt.
 \end{aligned}$$

Remplaçant (1.5) dans (1.6) on obtient (1.4).

Supposons

$$(1.7) \quad \eta^T F(\tau) \eta > 0, \quad \eta^T \dot{F}(\tau) \eta < 0, \quad \eta^T \ddot{F}(\tau) \eta \geq 0,$$

$\forall \eta$ - matrix 2×1 .

Si

$$(1.8) \quad 2\dot{F}(0)f(t) + \int_0^T \ddot{F}(|t-\tau|)f(\tau) d\tau = 0, \quad \forall t \in [0, T],$$

alors

$$f \equiv 0, \quad 0^T = [0, 0].$$

Multipliant (1.8) à gauche par $f^T(t)$, intégrant de 0 à T tenant compte de (1.4), on obtient

$$(1.9) \quad -\frac{1}{2} \int_0^T \int_0^T (f(t)-f(\tau))^T \ddot{F}(|t-\tau|) (f(t)-f(\tau)) d\tau dt + \\ + \int_0^T f^T(t) (\dot{F}(t) + \dot{F}(T-t)) f(t) dt = 0.$$

De (1.9) et (1.7) résulte

$$f \equiv 0.$$

Théorème. Si $S = \{p \in C^1[0, T]; p(0) = 0, p(T) = p_T \neq 0\}$, alors il n'existe pas de déformation p pour laquelle la fonctionnelle $W(p)$ soit minime.

Démonstration. De (1.3) résulte

$$(1.10) \quad W(p + \alpha p') = \int_0^T \int_0^t (\dot{p}^T + \alpha \dot{p}'^T(t)) F(t-\tau) (\dot{p}(\tau) + \alpha \dot{p}'(\tau)) d\tau dt$$

et tenant compte que

$$\delta W(p)[p'] = \frac{d}{d\alpha} W(p + \alpha p')|_{\alpha=0},$$

où $p'(t)$ — arbitraire et $p'(0) = p'(T) = 0$, on obtient

$$(1.11) \quad \delta W(p)[p'] = - \int_0^T p'^T(t) \left\{ \int_0^t \ddot{F}(t-\tau) \dot{p}(\tau) d\tau - \right. \\ \left. - \int_t^T \ddot{F}(\tau-t) \dot{p}(\tau) d\tau \right\} dt.$$

Du fait que $p'(t)$ est arbitraire, du optimum de (1.11) il résulte

$$(1.12) \quad \int_0^T \dot{F}(t-\tau) \dot{p}(\tau) d\tau - \int_0^T \dot{F}(\tau-t) \dot{p}(\tau) d\tau = 0.$$

Dérivant (1.12) par rapport à t , nous avons

$$(1.13) \quad 2\dot{F}(0)\dot{p}(t) + \int_0^T \ddot{F}(|t-\tau|)\dot{p}(\tau) d\tau = 0,$$

et selon (1.8) il résulte

$$\dot{p}(t) = 0 \Rightarrow p(t) = C.$$

Mais $p(0)=0$, $p(T)=p_T \neq 0$ et par conséquent on abouti à une contradiction. Donc il n'existe pas de déformation dans S qui minimise la fonctionnelle W .

3. L'existence dans L_2 . Soit

$$(2.1) \quad p(0)=0, \quad p(T)=p_0.$$

Les équations constitutives (1.1) deviennent

$$(2.2) \quad \begin{aligned} \sigma(t) &= A(0)\gamma(t) + B(0)x(t) + \int_0^t \dot{A}(t-\tau)\gamma(\tau) d\tau + \int_0^t \dot{B}(t-\tau)x(\tau) d\tau, \\ m(t) &= B(0)\gamma(t) + C(0)x(t) + \int_0^t \dot{B}(t-\tau)\gamma(\tau) d\tau + \int_0^t \dot{C}(t-\tau)x(\tau) d\tau. \end{aligned}$$

La relation (1.3), tenant compte de (1.2), (2.2), prend la forme

$$(2.3) \quad W(p) = p^T(t) \frac{F(0)}{2} p(t) \Big|_0^T + \int_0^T \int_0^t \dot{p}^T(t) \dot{F}(t-\tau) p(\tau) d\tau dt.$$

Intégrant par parties et tenant compte de (2.1) nous avons

$$(2.4) \quad \begin{aligned} W(p) &= p_0^T \frac{F(0)}{2} p_0 + p_0^T \int_0^T \dot{F}(t-\tau) p(\tau) d\tau - \int_0^T p^T(t) \dot{F}(0) p(t) dt - \\ &\quad - \frac{1}{2} \int_0^T \int_0^T p^T(t) \ddot{F}(|t-\tau|) p(\tau) d\tau dt. \end{aligned}$$

Entre (1.3) et (2.4) il existe une différence $p \in L_2(0, T)$ (dans (2.4)). On peut vérifier que (2.4) est une fonctionnelle continue par $L_2(0, T)$. Soit une déformation arbitraire $p \in L_2(0, T)$.

Vu que S est dense dans $L_2(0, T)$, il existe la suite $p_n \in S$, de sorte que $p_n \rightarrow p$ dans $L_2(0, T)$ pour laquelle $W(p_n) \rightarrow W(p)$.

T h é o r è m e. *Il existe une déformation unique $p \in L_2$ pour laquelle $W(p)$ est minime.*

Si on note

$$(2.5) \quad M = p_0^T \frac{F(0)}{2} p_0, \quad L(p) = p_0^T \int_0^T \dot{F}(T-t)p(t) dt,$$

$$Q(p, p) = - \int_0^T p^T(t) \dot{F}(0) p(t) dt - \frac{1}{2} \int_0^T \int_0^T p^T(t) \ddot{F}(|t-\tau|) p(\tau) d\tau dt,$$

la relation (2.4) devient

$$(2.6) \quad W(p) = M + L(p) + Q(p, p).$$

Mais $L(p)$ est une forme linéaire et $Q(p, p)$ est une forme bilinéaire et symétrique. De (2.5)₃, (1.4), (1.7) résulte que

$$(2.7) \quad Q(\beta, \beta) \geq 0, \quad \forall \beta \in C[0, T],$$

et donc par continuité $\forall \beta \in L_2(0, T)$.

On constate que

$$W(p+\beta) - W(p) = Q(\beta, \beta) + 2Q(p, \beta) + L(\beta).$$

Par conséquent, la condition nécessaire et suffisante pour que $\beta \in L_2(0, T)$ soit optimal, est que

$$(2.8) \quad 2Q(p, \beta) + L(\beta) = 0, \quad \forall \beta \in L_2(0, T),$$

où

$$(2.9) \quad \int_0^T \beta^T(t) \left[\dot{F}(T-t)p_0 - 2\dot{F}(0)p(t) - \int_0^T \ddot{F}(|t-\tau|)p(\tau) d\tau \right] dt = 0,$$

d'où résulte

$$(2.10) \quad \dot{F}(T-t)p_0 = 2\dot{F}(0)p(t) - \int_0^T \ddot{F}(|t-\tau|)p(\tau) d\tau = 0, \quad \forall t \in [0, T].$$

Ce système (équation) Fredholm a une solution unique si l'équation homogène attachée à l'équation (2.10) ($p_0=0$) a une solution banale (conformément à

l'alternative de Fredholm). Mais, l'équation homogène attachée à l'équation (2.10) est justement (1.8), qui conduit à

$$p(t) \equiv 0, \quad \forall t \in [0, T].$$

Par conséquent, il existe une solution unique $p(t) \in L_2(0, T)$ pour l'équation (2.10) de sorte que $W(p)$ est minime.

Reçue le 20 Février, 1991

Université Technique "Gh. Asachi", Jassy,
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DEFORMAREA OPTIMĂ ÎN TEORIA VÂSCOELASTICITĂȚII LINIARE MICROPOLARE

(Rezumat)

Se arată că există o deformare în L_2 pentru care o anumită funcțională devine minimă. Deformarea care minimizează respectiva funcțională se numește deformare optimă.

**DIPOLAR RELAXATION CHARACTERISTIC PARAMETERS
DETERMINATION BY THERMALY STIMULATED DISCHARGE
METHOD FOR STAR POLY [γ -STEARYL- α , L-GLUTAMATE]**

II. ALFA-TYPE MAXIMUM ANALYSIS

BY

**RODICA NEAGU, EUGEN NEAGU, I. I. NEGULESCU
and WILLIAM H. DALY**

1. Introduction. Thermally Stimulated Discharge Current (TSDC) technique is applied to the study of star poly [γ -stearyl- α , L-glutamate], star PSLG, which is a class of side-chain liquid-crystal polymer (LCP) that contains a highly dipolar dye. It is known that the TSDC thermograms are very sensitive to a number of LCP properties including glass transition, degree of mesogenic alignment and the amount of polarization that can be stored.

We have observed a good correlation between some properties and the results of DSC spectra. The TSDC technique is able to define the optimum conditions for poling these LCP systems and also it can be used to determine a number of fundamental properties such as activation energy, molecular relaxation distribution and polymer flexibility and, allows fundamental examination of the relationships between main chain and mesogenic motions.

2. Basic TSDC Experiment. Discussion. The basic result of a TSDC experiment is the thermogram showing the position of maximum current flow and this is related to polymer relaxations.

It has been shown by many authors that the relaxation processes in polymers can be described by a Debye model of relaxation only in the case when an appropriate distribution function is introduced. However, no experimental evidence exists regarding which of the parameters describing the Debye model is distributed. Usually it is assumed that either the activation

energy or natural frequency is distributed, and the shape of the distribution function is then calculated [1].

Table 1

Nr.	b °C/ /min	U_p V	T_p °C	t_p h	$j_{\max}$ 10^{-10} A/m ²	$T_{\max}$ °C	Shoulders		T_1 °C
							α_1 T_m °C	α_2 T_m °C	
1	2	30	20	7	43	44	19	-	54
2	1	30	20	14	34.3	49	15	-	58
3	3	30	20	7	38	37	-	-	44
4	3	30	20	20	94	40	-	-	54
5	3	62	20	6	124	45	19	-	55
6	0.82	62	20	240	104	37	7	-	43
7	0.73	80	20	20	75	43	17	-	50
8	0.8	83	20	20	70	37	13	-	49
9	1.33	83	20	19	106	37	7	-	51
10	0.83	83	20	100	172	42	18	-	51
11	2.4	83	20	20	170	45	-	-	55
12	3.2	83	20	52	338	49	35.5	-	58
13	2.2	83	20	20	141.6	40	31.34	-	46.5
14	1.3	83	48	22	1250	57.5	37	49.2	61
15	1.8	83	37	20	1172.6	54.5	35	46	63
16	1.3	25	30	20	289	45	-37	9	50
17	1.6	25	35	27	339	46	-39	7	52
18	1.6	25	22	14	193	43	-34	19	52
19	1.4	30	22	20	331	45	-34	7	53
20	1.2	30	20	20	246	41	-36	7	49

The thermograms obtained by us have two peaks which appear in all experiments indifferent of the conditions in which the TSDC are obtained. The first peak (the β type peak) was studied in another paper [4]. The second peak, the α -type peak, is more intense than the first and appears in the temperature range from 37°C to 57°C.

To put in evidence the mechanisms which produce this maximum we had realized more thermograms for each sample. Three samples of different thickness have been studied. The sample have been obtained from solution in toluen of the star PSLG. The solution have been poured on aluminium discs. After solvent evaporation a second aluminium electrode was deposited by vacuum evaporation.

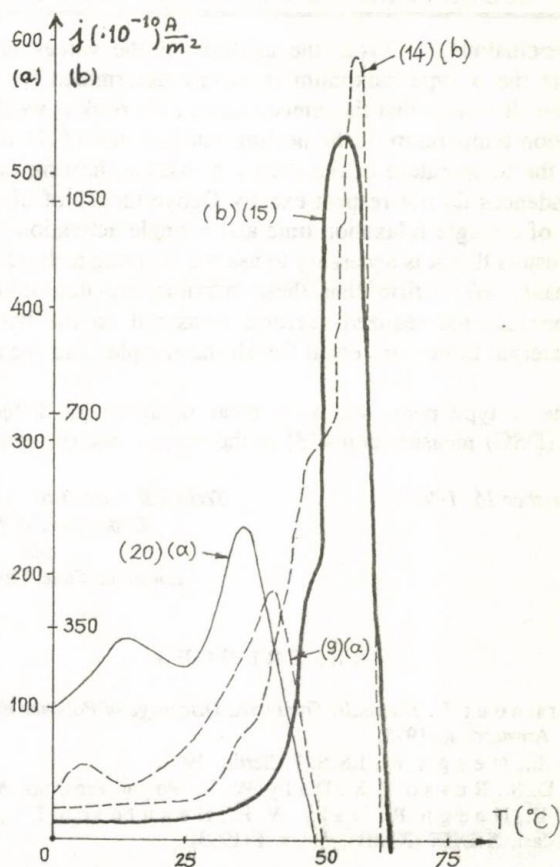


Fig. 1

In Fig. 1 are shown four thermograms numbered like in Table 1. The main parameters for this peak are presented in Table 1. A close examination of these parameters shows that in contrast with β peak, the α peaks contain one or two shoulders which are designated in Table 1 by α_1 and α_2 . The temperature at which these shoulders appears is different and some of the peaks correspond with the glass-rubber transition temperature T_g , which was determined by us in another paper [2]. This modification of the peak temperature can be determined by the weak uniformity of the arms of star PSLG that at this stage of the production of this polymer appears to be poor [3].

3. Conclusions. 1. From the analysis of the values in Table 1 it is observed that the α type maximum is mainly determined by dipolar charge dezorientation. It is seen that the temperature of the peak is weak dependent on the polarization temperature if the heating rate is constant. In the same time it is seen that the temperature of the peak increases as heating rate b increases. These dependences do not respect exactly Debye theory of dipolar relaxation for the case of a single relaxation time and a single activation energy.

2. It results that it is necessary to use the cleaning technique for the study of these peaks. We affirm that these maxima are determined by dipolar relaxation because the realized currents measured on the both sides of the sample in external circuit are equal for all the samples and for all temperature range.

3. The α type peak was very clear obtained by differential scanning calorimetry (DSC) measurements [3] in the same range of temperatures.

Received December 16, 1992

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DETERMINAREA PRIN METODA DESCĂRCĂRII STIMULATE TERMIC A PARAMETRILOR CARACTERISTICI RELAXĂRII DIPOLARE PENTRU POLY [γ -STEARIL- α , L GLUTAMAT] STELAT

II. Analiza maximelor de tip alfa

(Rezumat)

Pentru PSLG stelat se stabilește că maximele de tip α sunt în cea mai mare parte determinate de dezorientarea sarcinii dipolare. Se conchide că aceste maxime se datoresc relaxării dipolare deoarece curenții mășurați pe ambele fețe ale eșantioanelor în circuitul exterior sunt egali, oricare ar fi eșantioanele și pentru tot intervalul de temperatură.

THE INFLUENCE OF THE IMPURITIES ON PHOTOELECTRODS PROPERTIES IN $\alpha\text{-Fe}_2\text{O}_3$

BY

EUGEN NEAGU and SERGIU ISTRATE

1. Introduction. The photoelectrolysis of water has been investigated by experiments on cells consisting of an illuminated *n*-type semiconductor anode, an aqueous electrolyte, and a platinized-Pt cathode. It has been found that such cells operate either in the photogalvanic mode (no H_2 evolved) or in the photoelectrolytic mode (H_2 evolved at the cathode by decomposition of water), depending on whether or not the electrolyte surrounding the cathode contains dissolved oxygen. In both cases, current flows through the external circuit and O_2 is evolved at the anode.

These results have aroused interest in the possible employment of photoelectrolysis for the large-scale utilization of solar energy to produce gaseous H_2 , a fuel that can be stored indefinitely and transported conveniently over long distances. In order to further evaluate the potential of the Fe_2O_3 for this application, we have been investigating the photoelectrolysis by experiments on cells with polycrystalline Fe_2O_3 impurified anodes.

2. Experimental Procedure. The sample were obtained by sintering the $\alpha\text{-Fe}_2\text{O}_3$ power impurified with ZrO_2 , TiO_2 and Nb_2O_5 respectively. The sintering temperature was 1200°C during 24 hours and the cooling was slow in the furnace. The compositions was $\alpha\text{-Fe}_2\text{O}_3$ impurified with 0.7; 0.4; 0.1 and 0.07 % respectively of ZrO_2 , TiO_2 and Nb_2O_5 respectively. In the first stage [1] we tested the compositions from samples of $\alpha\text{-Fe}_2\text{O}_3$ impurified with 1; 0.1 and 0.01 % respectively and the best results was obtained with 0.1 % impurities; then we investigated in detail the range about this composition.

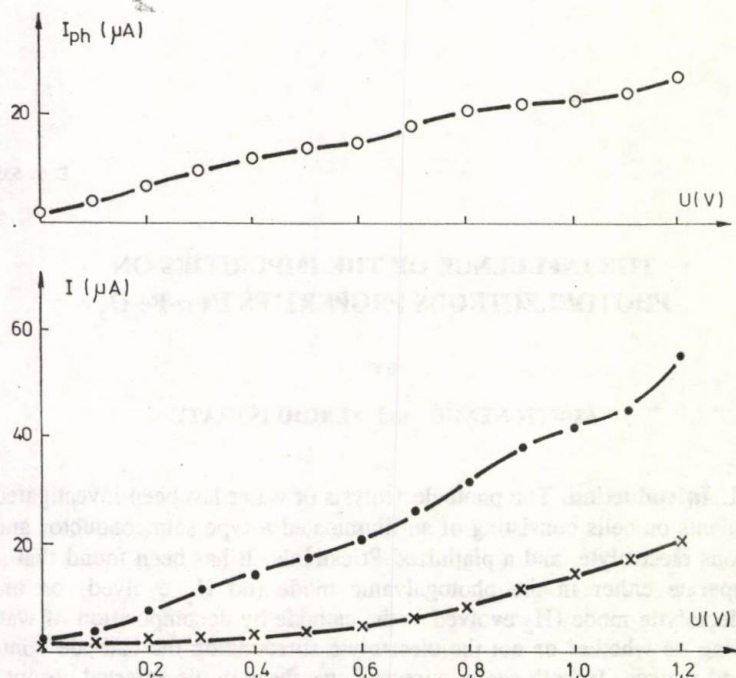


Fig. 1 — I_d and I_{ph} respectively, versus external potential for the sample with 0.1 % ZrO_2 .

In the present work are discussed the current-potential curves obtained by potentiostatic method. The work electrode (the photoanodes of $\alpha\text{-Fe}_2\text{O}_3$ impurified), counter electrode (the platinum cathode) and reference electroed (saturated calomel electroed—SCE) were placed in a quartz vessel. We tested many electrolytes and the 2M NaOH was choosed. The reference potential between work electrode and SCE was determined by a high input impedance electronic voltmeter. The work electrode was illuminated by a 200 Watt vapour mercury lamp. It was determined the dark current intensity and light current intensity respectively, versus external potential and reference potential, respectively.

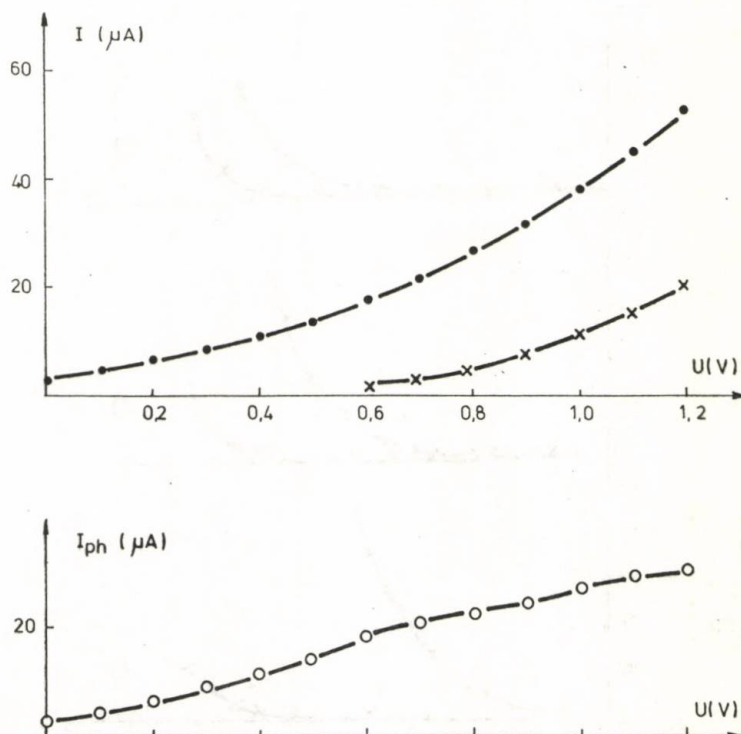


Fig. 2 — I_l , I_d and I_{ph} respectively, versus external potential for the sample with 0.07 % ZrO_2 .

The electrical contact on work electrode was InGa on one side and electrolyte on the other side. The lighting flux was maintained constant.

3. Results and Discussion. In the Figs. 1 and 2 we present the dependence of external potential for the light current (the work electrode is illuminated), the dark current and their difference ($I_l - I_d = I_{ph}$) named photocurrent in the samples impurified with 0.1 % and 0.07 % of ZrO_2 . These measurements were repeated and reproducible.

In general, the value of 1.2 V for external potential was not exceeded from two reasons:

- a) the photoelectrolysis is interesting only when the applied external

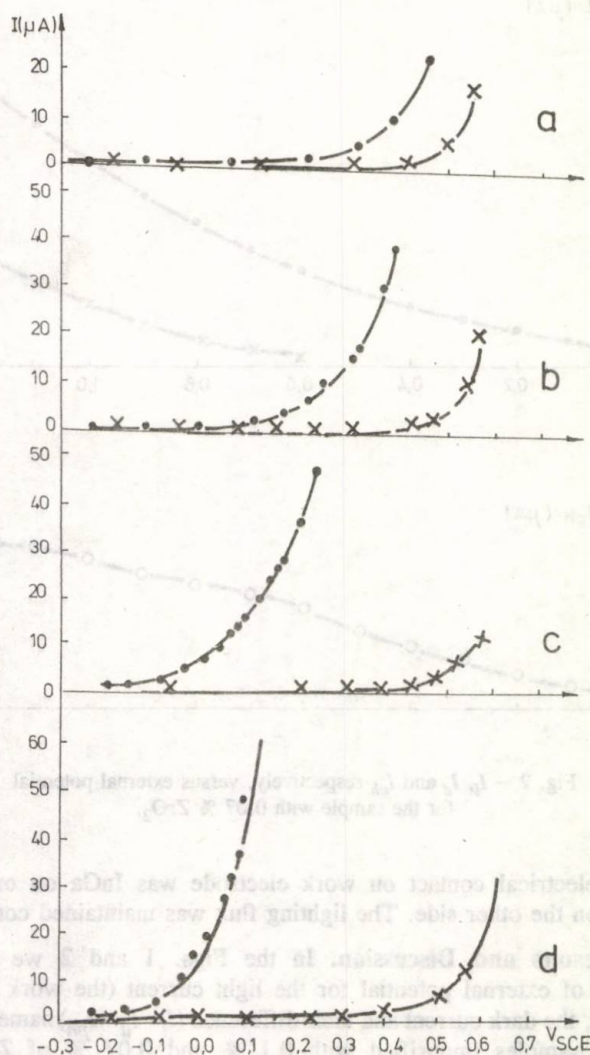


Fig. 3 — I_l and I_d respectively, versus reference potential for samples impurified with ZrO_2 : a) 0.7 %; b) 0.4 %; c) 0.1 %; d) 0.07 %.

potential is smaller than the decomposition voltage for water by electrolysis i.e. 1.48 V;

b) when large external voltage are utilized, for some photoelectrodes could appear irreversible processes and the results interpretation are more difficult.

In Fig. 3 we present the light current and dark current, respectively, versus potential between work electrode and SCE. Also these measurements are reproducible. All determinations were carried out on the $\alpha\text{-Fe}_2\text{O}_3$ impurified with ZrO_2 , but the behaviour of $\alpha\text{-Fe}_2\text{O}_3$ impurified with TiO_2 and Nb_2O_5 , respectively, are similar.

We carried out these two types of measurements because we consider that they are very important for the semiconductors utilization as photoelectrodes. The first type presents the importance from the point of view to obtain assisted photoelectrolysis with semiconductors anodes. The second gives us many informations both about the phenomena generated at the semiconductor-electrolyte interface and on the potential distribution in the photoelectrochemical cell.

In alls figures we noted the currents by: • — light current; × — dark current and ○ — photocurrent respectively. The corresponding ZrO_2 impurity is:

Sample	10	11	12	13
%	0.7	0.4	0.1	0.07

We repeated the measurements after a long time interval (months) on the same sample and observed a good repeatability; the photoelectrodes have a good chemical stability.

We observe that the photocurrent increases with the external applied potential and reaches a saturation value when all electron-hole pairs are efficiently separated.

For the obtaining of photoelectrodes with optimum properties it is necessary to realize a compromise from the point of view of the donor impurities concentration in the $\alpha\text{-Fe}_2\text{O}_3$. On the one hand the impurity level must be larger because the resistivity must be small; on the other hand the level must be smaller and so the space charge depletion region is large and then the separation of the carriers is efficient. Moreover when the impurities concentration increases, results a hole concentration decreasing, hence a decreasing of the photopotential.

Received December 17, 1991

Technical University "Gh. Asachi", Jassy,
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INFLUENȚA IMPURITĂȚILOR ASUPRA PROPRIETĂȚILOR
DE FOTOELECTROD ALE COMPUSULUI $\alpha\text{-Fe}_2\text{O}_3$

(Rezumat)

Se analizează comportarea ca fotoelectrod a compusului $\alpha\text{-Fe}_2\text{O}_3$ impurificat cu ZrO_2 , TiO_2 și respectiv cu Nb_2O_5 .

THE MOBILITY OF ELECTRONS IN $\alpha\text{-Fe}_2\text{O}_3$ IMPURIFIED WITH NIOBIUM

BY

SERGIU ISTRATE and MARIANA ISTRATE

1. Introduction. The ratio of the drift velocity v_d to the electric field intensity E is termed carrier mobility

$$(1) \quad \mu = \frac{v_d}{E} = \frac{q\tau}{m},$$

where q is the charge carrier, τ — the relaxation time of scattering processes and m — the carrier's effective mass.

The specific conductance of a semiconductor is

$$(2) \quad \sigma = qn\mu,$$

where n is the electron concentration. Then we can determine the mobility of the carriers when we know the concentration carriers, their charge and the conductivity of the material.

From the theoretical treating of the Seebeck phenomenon [1]...[3] it results a relation between Q — Seebeck coefficient and E_F — energy Fermi level, in a simplified manner, which is justified from the materials with narrow additional energy bands (donor bands or acceptor bands) or, much more, the levels are localized, when the term due to the kinetic energy who promotes Seebeck effect is negligible [4]; the relation is

$$(3) \quad Q_n = -\frac{E_c - E_F}{eT},$$

where E_c is the bottom level of the conduction band, e — electron charge and

T — absolute temperature. This relation is frequently utilized in the works about Seebeck phenomenon [4]...[10]. Then $Q_n T = E_c - E_F$ (eV).

K i r e e v [1] presents the formula for density of the electrons in a nondegenerate semiconductor

$$(4) \quad n = N_c \exp \left[- \frac{E_c - E_F}{kT} \right],$$

where N_c is the effective number of states for conduction band who depends of the temperature and the carrier's effective mass.

M o r r i n [4] considered that for ferric oxide, N_c is constant and equal to the number of atoms in the volume unit. We determine N_c with the aid of the volume unit cell, which is $100.5 \times 10^{-24} \text{ cm}^3$ [11] and results $N_c = 4 \times 10^{28} \text{ m}^{-3}$; then we calculate n — the relation (4), and with the experimental determined σ we deduce μ — by formula (2).

2. Experimental Method. The $\alpha\text{-Fe}_2\text{O}_3$ samples were obtained by the sintering of the pure oxides $\alpha\text{-Fe}_2\text{O}_3$ and Nb_2O_5 . The powders with chemical composition of 0.7 (N_1); 0.4 (N_2); 0.1 (N_3) and 0.07 (N_4) at % respectively was mixed in a wett ball mill for two hours, and then the mixture water-powder was at once dried in a drying closet. The mixed and dried powder was pressed with the aid of a hydraulic press, in the form of a rectangular prism. During the pressing in pre-established places were introduced four platinum wires for good electrical contacts. The sintering temperature was comprised between $1200 \dots 1300^\circ\text{C}$ and the duration of the bearing was $6 \dots 24$ hours. The cooling in air was slow in the furnace. X-ray diffraction studies (DRON-2,0 diffractometer, utilising $\text{FeK}_{\alpha\beta}$ radiation) have been undertaken to confirm the structure $R\bar{3}c$; other phases in the sintered materials were not observed.

The most accurate determinations for electrical properties in semiconductors are obtained when we utilize a rectangular sample with the lateral arms named "bridge" [9], [12], [13]. We realized samples with four arms: two used for the current passing, and the other lateral two for the voltage drop determination.

We designed and carried out a measuring equipment presented in [4] with which we succeeded to determine electrical properties in the temperature range $300 \dots 1350 \text{ K}$.

With a view to study Seebeck coefficient Q , we measured the electromotive force between the two lateral arms of the sample placed in a temperature gradient $5 \dots 15 \text{ K}$ available on the furnace axis.

With a view to obtain the electrical conductivity σ we establish a current through the probe and determine the voltage drop. Then we deduce R and from the sample geometry we calculate σ . Because σ varies strongly with the

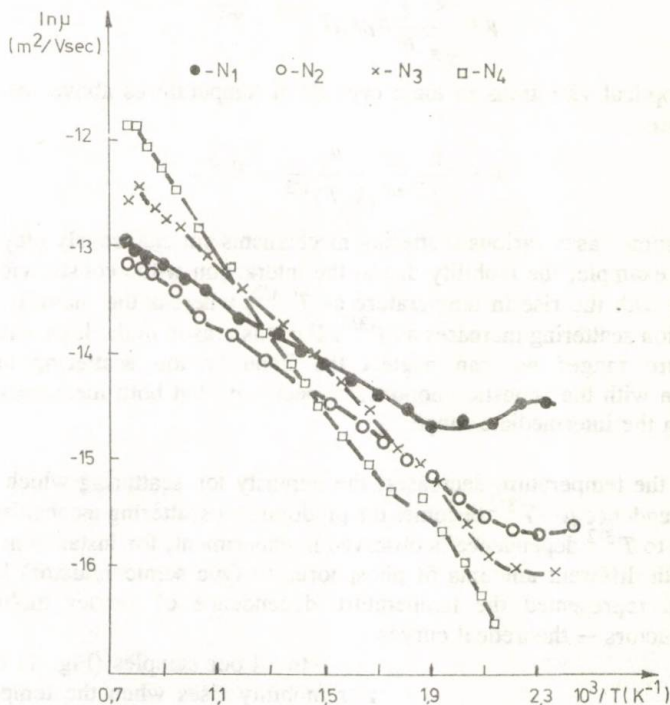


Fig. 1 — The carrier mobility versus temperature in $\alpha\text{-Fe}_2\text{O}_3$ impurified with niobium.

3. Results and Discussion. The symbols of the samples are: N_1 for $\alpha\text{-Fe}_2\text{O}_3$ impurified with 0.7 % Nb; N_2 — 0.4 %; N_3 — 0.1 % and N_4 — 0.07 % respectively.

In the Fig. 1 is presented the variation of the carrier mobility versus temperature. The dependence of the carrier mobility on temperature is determined by the prevalent scattering mechanisms at a given temperature. This subject is treated in [1], [2], [15]. A n s e l m [2] calculated the mobilities due to different kinds of the scattering processes:

a) acoustic vibrations

$$(5) \quad \mu = \frac{4}{3\sqrt{\pi}} \frac{e}{m} \frac{a_A}{(k_0 T)^{3/2}} \sim T^{-3/2};$$

b) impurity ions

$$(6) \quad \mu = \frac{8}{\sqrt{\pi}} \frac{e}{m} a_I (k_0 T)^{3/2} \sim T^{3/2};$$

c) optical vibrations in ionic crystals at temperatures above the Debye temperature

$$(7) \quad \mu = \frac{8}{3\sqrt{\pi}} \frac{e}{m} \frac{a_p}{(k_0 T)^{1/2}} \sim T^{-1/2}.$$

In some cases various scattering mechanisms simultaneously play a vital part. For example, the mobility due to the interaction with acoustic vibrations decreases with the rise in temperature as $T^{-3/2}$, whereas the mobility due to impurity ion scattering increases as $T^{3/2}$. For this reason in the high- and low-temperature ranges we can neglect the impurity ion scattering and the interaction with the acoustic phonons, respectively, but both mechanisms will be vital in the intermediate range.

As the temperature decreases, the impurity ion scattering which results in the dependence $\mu \sim T^{3/2}$ becomes the predominant scattering mechanism. The transition to $T^{3/2}$ dependence is observed in experiment, for instance in silicon doped with different amounts of phosphorus (*n* type semiconductor) [15]. In Fig. 2 is represented the temperature dependence of carrier mobility in semiconductors — theoretical curves.

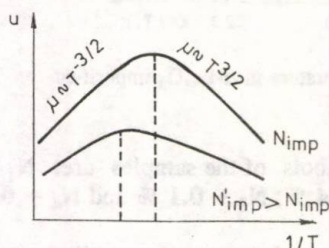


Fig. 2 — Theoretical curves from temperature dependence of the carrier mobility at different impurity concentrations.

In all our samples (Fig. 1) the carrier mobility rises when the temperature increases, and in the case of small doped probes (*n* — small) at relative high temperatures, we notice a diminution, namely the influence of acoustic phonons is notable. The fact is that the mobility in the small doped samples at low temperatures is more smaller than in the doped ones and, at medium temperatures, is more large. According to the Fig. 2, normally is that the mobility due to ion impurity scattering be smaller at high concentration of impurities. From these ascertainments we can deduce that the influence of the grain

boundary is stronger in the smaller doped probes.

4. Conclusions. 1. In the $\alpha\text{-Fe}_2\text{O}_3$ impurified with niobium at high temperatures the scattering of electrons by the acoustic phonons is notable.

2. At low temperatures the influence of the grain boundary is stronger in the smaller doped probes.

Received December 17, 1992

Technical University "Gh. Asachi", Jassy,
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MOBILITATEA ELECTRONILOR ÎN $\alpha\text{-Fe}_2\text{O}_3$ IMPURIFICAT CU NIOBIU

(Rezumat)

Pe probe de $\alpha\text{-Fe}_2\text{O}_3$ sinterizat, dopate la nivele relativ mici cu Nb_2O_5 , se determină mobilitatea purtătorilor în intervalul de temperatură 300...1350 K. Se discută dependența mobilității purtătorilor de temperatură comparând-o cu dezvoltarea teoretică a acestui subiect și respectiv cu rezultatele obținute pe alte materiale.

the same time, the rate of polymerization is also affected by the concentration of the monomer.

Received December 17, 1967
Journal of Polymer Science, Part A: General Papers
Department of Chemistry

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MOBILITY ELECTRONICS IN POLYMER
SOLIDS

(Received)

The mobility of electrons in polymers is a subject of great interest to many scientists. It is well known that the mobility of electrons in polymers is much lower than in inorganic solids. This is due to the fact that the electrons in polymers are localized on the polymer chains and are not free to move through the material.

INFLUENCE OF THE TiO_2 AND ZrO_2 ADDITION OVER SOME MAGNETIC AND ELECTRIC PROPERTIES OF ONE RADIOFREQUENCY FERRITE

BY

DORIN CONDURACHE, N. FOCA, EMIL LUCA and EUGEN OSADET

1. Introduction. It is known that the iron deficient nickel ferrite exhibits the permeability dispersion phenomenon under high frequency [1], [2]. The 49.7 % Fe_2O_3 — 50.3 % NiO ferrite studied in [2] having a low magnetic permeability is practically unuseful as magnetic core.

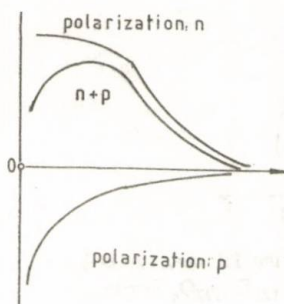


Fig. 1 — Dependence of the dielectric polarizability on frequency evidencing the contribution of two polarization mechanisms.

In order to increase its permeability, at an operating temperature up to $80\ldots 100^\circ\text{C}$, a Ni-Zn-Cu ferrite was chosen. The presence of the Cu^{2+} ions, partially converted to Cu^+ at sintering temperature higher than 900°C , is responsible for the appearance of one type p electric conduction, on negative ions of oxygen [3]...[7]. The low mobility of the type p carriers leads to a characteristic variation of the polarization against frequency [3], as shown in Fig. 1.

Due to these reasons, a ferrite with chemical composition of 66.1% Fe_2O_3 , 17.9% ZnO , 12.0% NiO and 4.0% CuO , namely $\text{Ni}_{0.384}\text{Zn}_{0.525}\text{Cu}_{0.121}\text{Fe}_{1.97}\text{O}_4$, has been studied.

2. Experimental Procedure. The ferrite of composition mentioned above was impurified under rigorous control during oxides grinding, with TiO_2 and ZnO_2 , respectively, over stoichiometry between 0.0 and 2.0 percent in weight.

The standard ceramic technology was employed for samples preparation. Sintering was carried out in air, at 1350° for 4 hours, followed by slow cooling.

The saturation magnetization was measured by means of one magnetometer with vibrating sample at liquid nitrogen temperature. The resistivity of samples was measured with a Wheatstone bridge. The initial permeability and magnetic and dielectric losses were determined with one admittances bridge using samples in the form of coiled tori. The Curie-point was measured from the temperature dependence of the initial permeability. The bridge measurements were performed at 50 kHz.

3. Results and Discussion. Additions of TiO_2 and ZrO_2 , respectively, in any amounts within the limits mentioned above, do not modify the Curie-point found to be at 242°C. Specific saturation magnetization σ , after a slow increase for little additions, decreases slowly (with about 12 percent) for greater ones (Fig. 2).

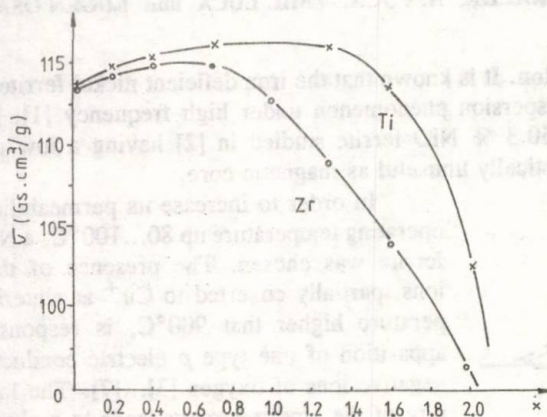


Fig. 2 — Dependence of the saturation magnetization on the TiO_2 (—x—x—x—) and ZrO_2 (—o—o—o—) addition in the $\text{Ni}_{0.384}\text{Zn}_{0.525}\text{Cu}_{0.121}\text{Fe}_{1.97}\text{O}_4$ ferrite.

As it comes out from Fig 2, decreasing tendency is more rapidly for ferrite with zirconium addition. In the case of the TiO_2 addition this phenomenon is due to the ilmenite (FeTiO_3) precipitation in the superficial layer of granules. The percentage of ferromagnetic substance inside the samples becomes lower, so that the ratio magnetization/sample volume decreases.

In the case of the ZrO_2 addition, a similar compound (FeZrO_3) possibly precipitates, having the same effect. The σ decrease taking place for additions of zirconium less than titanium ones is probably due to its lower solubility in

the spinel lattice than that of titanium. Indeed, the ionic radius of Ti^{4+} being $0.61\text{\AA}$ and that of Zr^{4+} $0.8\text{\AA}$, explains why zirconium is less soluble in the spinel lattice than titanium.

The initial permeability of the investigated ferrite is much slower influenced by little additions, in comparison with that of the Mn-Zn ferrite reported in cited literature [8], [9].

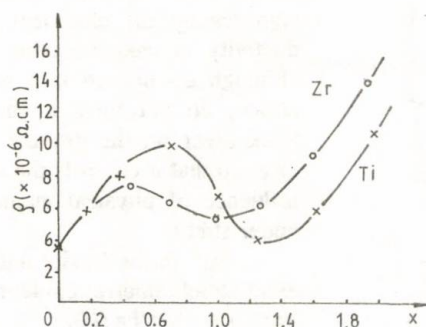


Fig. 3 — Variation of the electrical resistivity in d.c. vs TiO_2 (—x—x—x—) and ZrO_2 (—o—o—o—) content.

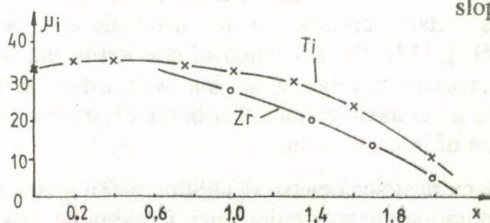


Fig. 4 — Dependence of the initial permeability on the TiO_2 (—x—x—x—) and ZrO_2 (—o—o—o—) content.

As the Mn-Zn ferrite contains excessive iron, the Ni-Zn-Cu ferrite under study is almost free of Fe^{2+} ions.

The presence of titanium in ferrite determines the decrease of the Fe^{+2} ions number by electron capture. If in the Mn-Zn ferrite [8]...[10] this phenomenon is possible with significant effects on permeability and resistivity, in the Ni-Zn-Cu ferrite this mechanism is negligible.

The increasing tendency of the initial permeability for small additions does not occur and, its decreasing for large additions is gently-sloping, due to the formation of $FeTiO_3$ and $FeZrO_3$ precipitates (Fig. 4).

Let us remark again the fact that the beginning of the initial permeability decrease is shown for less additions of Zr than Ti. Also, it is to be noted that the additions at which the curve shape is changed correspond to those presented in Figs. 2 and 3.

The dependence of both relative factor of losses $\tan \delta / \mu_i$ and factor of losses by relative hysteresis, h / μ_i^2 — determined under 50 kHz, on the TiO_2 and ZrO_2 additions, is represented in Fig. 5.

A decrease of losses by Foucault currents can be seen, that becomes negligible for addition larger than 1.9% TiO_2 and 1% ZrO_2 , respectively, as well as a significant increase of the losses by hysteresis for the same additions.

All these facts suggest an intensive segregation of dielectric and nonmagnetic compounds at the ferrite granules surface, starting with a certain addition of TiO_2 or ZrO_2 . The value of the limit concentration up to which TiO_2 or ZrO_2 are still soluble in lattice is about 1.4 % and 1.2 % for TiO_2 and ZrO_2 , respectively. In the solubility limit, the Ti^{4+} and Zr^{4+} ions go into

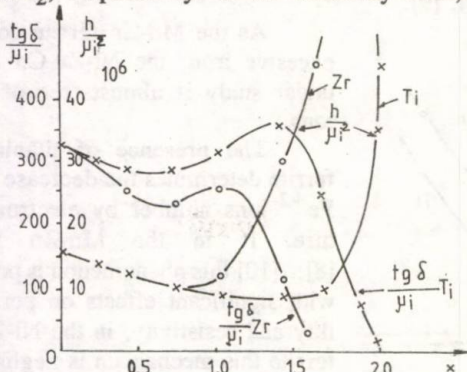


Fig. 5 — Dependence of the hysteresis and Foucault currents losses on TiO_2 (—x—x—x) and ZrO_2 (—o—o—o) content.

lattice, but do not influence significantly the electrical conductivity or magnetization, or, although do not go into spinel lattice, do not form a enough thick layer on the granule surface, so that it controls the great influence of physical quantities under study.

But, in the ferrite with an extra stoichiometric content of iron, i.e. $\text{Me}/\text{Fe} \leq 0.5$, is possibly and probably that the effect of small additions of TiO_2 or ZrO_2 to be more important.

Finally, we most remember that the replacing of the Zn

ions by Cd ones, diminishes certain variations of any magnetic and electric parameters vs. composition [11], [12]; the utilization of one ion in the site of the other, with the same electronic configuration, but with different ionic radius, induces principally the same modifications, but in the crystalline lattice will be better tolerated the ion of inferior radius.

4. Conclusions. 1. The extra stoichiometrical addition of TiO_2 and ZrO_2 influences negligibly the saturation magnetization and Curie-point. By the entered tetravalent ions, the amount of divalent ions of iron, that can participate at electronic conduction, is diminished, some of them passing to Fe^{3+} .

2. The electric conductivity and magnetic losses decrease significantly as effect of modification of the ratio $\text{Fe}^{2+}/\text{Fe}^{3+}$ in favour of the last one.

3. There is a maximum solubility limit of the Ti^{4+} and Zr^{4+} ions in ferrite, depending on ferrite type employed, being less for zirconium, which has the ionic radius greater than titanium.

Received August 2, 1992

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INFLUENȚA IMPURIFICĂRII CU TiO_2 ȘI ZrO_2 ASUPRA
UNOR PROPRIETĂȚI MAGNETICE ȘI ELECTRICE
ALE UNEI FERITE DE RADIOFRECVENȚĂ

(Rezumat)

Se studiază influența adaosurilor suprastoichiometrice de TiO_2 și respectiv ZrO_2 într-o ferită cu compoziția $\text{Ni}_{0.384}\text{Zn}_{0.525}\text{Cu}_{0.121}\text{Fe}_{1.97}\text{O}_4$. Se constată o influență redusă a adaosurilor asupra magnetizării la saturație și a temperaturii Curie, precum și o scădere remarcabilă a conductibilității electrice și a pierderilor magnetice, o dată cu creșterea concentrației de impurități. Se găsește o concentrație limită de solubilitate a ionilor Ti^{4+} și Zr^{4+} în rețeaua spinelică, mai mică în cazul zirconiului.

THE MAGNETIZATION OF $\text{Fe}_{40}\text{Ni}_{40}\text{B}_{20}$ AMORPHOUS RIBBONS IN ORTHOGONAL MAGNETIC FIELDS

BY

GH. CĂLUGĂRU, C. MUNTEANU, I. APACHIȚEI,
SORIN ISTRATE and I. RUSU

1. Introduction. In the literature [1] some results about hysteresis magnetic characteristics have been presented for ribbons of different amorphous alloys magnetized in unidirectional field, but there are no references on magnetization of these alloys in orthogonal magnetic fields.

The shape of the magnetization curve obtained for ferromagnetic compounds in two orthogonal magnetic fields, one alternating H_a and the other H_s , depends on the detecting direction of the magnetic flux [2].

In this paper preliminary results are presented on the magnetization of $\text{Fe}_{40}\text{Ni}_{40}\text{B}_{20}$ amorphous ribbons in orthogonal magnetic fields — H_a and H_s — both of them included in the ribbon plane, when the magnetic flux detection is made in the direction of the H_a field, for different angles between this direction and the ribbon axis.

Experimental results proved that the shape of the hysteresis loops depends both on the angle made by the ribbon axis with the alternating field direction and the value and sense of the static magnetic field. The existence of an unidirectional anisotropy, with the easy magnetization axis on the direction of ribbon formation is noticed.

2. Experimental. $\text{Fe}_{40}\text{Ni}_{40}\text{B}_{20}$ amorphous ribbons with a $2 \text{ mm} \times 0.025 \text{ mm}$ rectangular cross section obtained by ultrarapid cooling of the melted alloy using a well known method [3] were prepared.

The H_s and H_a orthogonal magnetic fields were created by attaching a

d.c. Helmholtz coils system to a 50 Hz a.c. Howling installation [4]. Hysteresis loops were observed on the display screen of a cathodic oscillograph.

3. Results and Discussion. The shape of the hysteresis loop depends on both the H_s magnitude and α , the angle made by the ribbon axis with H_a (Fig. 1).

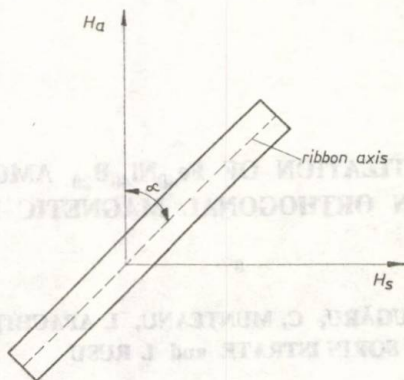


Fig. 1 — The position of the ribbon in relation to the H_a and H_s magnetic fields.

Fig. 2 shows the hysteresis loops for three different values of the α angle when the alternating magnetic field is maximum, $H_{am}=378$ A/m and the static one is absent, $H_s=0$.

The shape observed at these loops indicates the presence of an unidirectional anisotropy with the easy magnetization axis directed along the ribbon.

This unidirectional anisotropy, also observed by other authors, from structural analysis of the magnetic domains [5] or from measurements of the torsion couple [6], was explained by the pair distribution of atoms during the solidification of the melt along its flow direction.

Another explanation for the existence of this unidirectional anisotropy is that during the cooling of the melt according to the method applied [3] a rolling process occurs along the ribbon as was observed at the cold rolling of Fe-Ni crystalline ribbons [7].

The influence of the H_s static field on the hysteresis magnetic characteristics is pointed out in Figs. 3, 4 and 5.

Figs. 3 and 4 show that maximum and remanent magnetizations, M_m and M_r , respectively, decrease quite markedly with the increase of the static magnetic field for specimens with $\alpha=0$. Consequently, the smaller the angle between the easy magnetization axis (ribbon axis) and H_s , the lower the effect of this field on the magnetization.

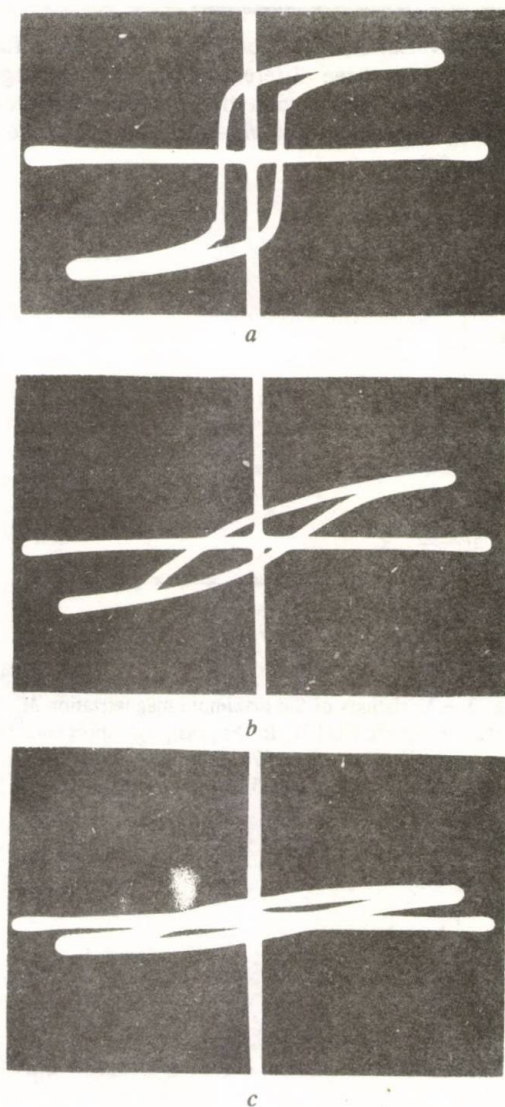


Fig. 2 — Hysteresis loops of the $\text{Fe}_{40}\text{Ni}_{40}\text{B}_{20}$ amorphous ribbons obtained when only the alternating magnetic field is present with the magnitude $H_{am}=378$ A/m:
a) $\alpha=0^\circ$; b) $\alpha=45^\circ$; c) $\alpha=90^\circ$.

In Fig. 5 the variation of the coercitive field H_c as a function of the static field H_s is presented for three different values of the α angle. It follows that, with the increase of H_s , the H_c coercitive field increases for every value of α at the $\text{Fe}_{40}\text{Ni}_{40}\text{B}_{20}$ amorphous ribbons, while for cold-rolled Fe-Ni crystalline ribbons decreases [7].

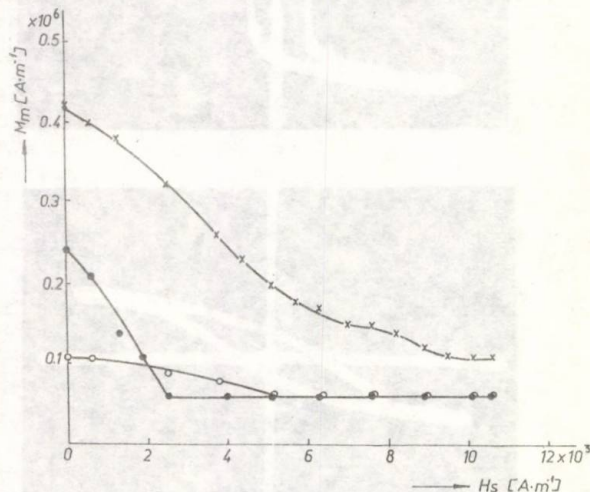


Fig. 3 — Variations of the maximum magnetization M_m with the static magnetic field H_s for $\text{Fe}_{40}\text{Ni}_{40}\text{B}_{20}$ amorphous ribbons when $H_{am} = 378 \text{ A/m}$: $\times$ — $\alpha = 0^\circ$; $\bullet$ — $\alpha = 45^\circ$; $\circ$ — $\alpha = 90^\circ$.

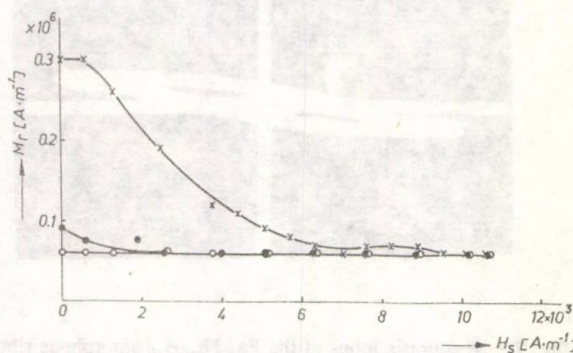


Fig. 4 — Variations of the remanent magnetization M_r with the static magnetic field H_s for $\text{Fe}_{40}\text{Ni}_{40}\text{B}_{20}$ amorphous ribbons when $H_{am} = 378 \text{ A/m}$: $\times$ — $\alpha = 0^\circ$; $\bullet$ — $\alpha = 45^\circ$; $\circ$ — $\alpha = 90^\circ$.

Comparing the shape and the characteristics of the hysteresis loops, obtained for $\alpha=90^\circ$ and $H_s=0$, to the ones obtained for $\alpha=0$ and large values for H_s , it may be stated that the presence of H_s is equivalent to an easy magnetization axis along the static field direction.

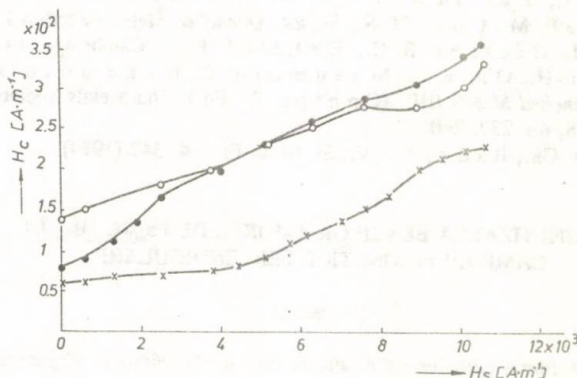


Fig. 5 —Variations of the coercive field H_c with the static magnetic field H_s for $\text{Fe}_{40}\text{Ni}_{40}\text{B}_{20}$ amorphous ribbons when $H_{am}=378 \text{ A/m}$:
 $\times$ — $\alpha=0^\circ$; $\bullet$ — $\alpha=45^\circ$; $\circ$ — $\alpha=90^\circ$.

4. Conclusions. 1. The shape of the hysteresis loops obtained for $\text{Fe}_{40}\text{Ni}_{40}\text{B}_{20}$ amorphous ribbons placed in orthogonal magnetic fields included in the ribbon plane depends on both the angle made by the ribbon axis with H_a alternating field direction and the magnitude of the H_s static field.

2. The presence of an unidirectional anisotropy with the easy magnetization axis along the ribbon is revealed and it may originate from the stresses occurred in the formation of the ribbon which is supposed to be subjected to a rolling process during its preparation.

3. Maximum and remanent magnetizations, M_m and M_r respectively, of the $\text{Fe}_{40}\text{Ni}_{40}\text{B}_{20}$ amorphous ribbons vary with the H_s static field in the same way as the Fe-Ni crystalline specimens, while the coercive field H_c varies contrarily.

Received February 25, 1993

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MAGNETIZAREA BENZILOR AMORFE DE $\text{Fe}_{40}\text{Ni}_{40}\text{B}_{20}$ ÎN CÂMPURI MAGNETICE PERPENDICULARE

(Rezumat)

Se studiază forma curbelor de histerezis obținute în câmpuri magnetice perpendiculare, unul alternativ și celălalt static, pentru benzile amorfe de $\text{Fe}_{40}\text{Ni}_{40}\text{B}_{20}$.

Rezultatele experimentale au arătat că forma curbelor de histerezis depinde de unghiul făcut de axa benzii cu direcția câmpului alternativ și de valoarea și sensul câmpului magnetic static. Se remarcă prezența unei anizotropii uniaxiale cu axa de ușoară magnetizare pe direcția de obținere a benzii.

RECENZII

BOOK REVIEWS

РЕЦЕНЗИИ

N.T. SHOREY and R. TIJDEMAN, *Exponential Diophantine Equations*. Cambridge University Press, Cambridge, 1986, 240 pp., ISBN 0-521-26826-5.

This excellent monograph deals with applications of estimates of linear forms in logarithms of algebraic numbers to prove the existence of effectively computable upper bounds for the magnitudes of the solutions of exponential diophantine equations. For the readers convenience the authors begin with chapters on algebraic number theory, on linear forms in logarithms and on recurrence sequences, which contain all the required preliminaries.

Ch. 1 deals with *Equations $x+y=z$ in Algebraic Integers* x, y, z from a fixed algebraic number field such that the ideal $[xyz]$ is composed of prime ideals from a given finite set. In Ch. 2, a remarkable *Consequence of Baker* (1972) is worked out, namely that for given non-zero integers A, B, C, D and under suitable conditions, the equation $Ax^m + By^m = Cx^n + Dy^n$ implies that m is bounded by an effectively computable number depending only on A, B, C and D . Ch. 3 deals with *Non-Degenerate Binary Recurrence Sequences*. In Ch. 4, *Recurrence Sequences of Higher Order* are investigated.

Chs. 5–8 concern *Polynomial Equations* and their p -adic analogues. Baker (1968), by way of his fundamental researches on linear forms in logarithms, gave an effective version of *True's Theorem* (Ch. 5). Consequently it was possible to give effective versions of earlier results on the *Solutions of Superelliptic Equations*, $f(x)=y^m$ where $m>2$ is a given rational integer and f is a given polynomial (Ch. 6). By the natural p -adic analogue of Baker's theory of linear forms in logarithms, it was possible to show in an effective way that *Equations $f(x, y)=k, f(x)=y^m$* have only finitely many solutions in rational numbers (Chs. 7 and 8).

Chs. 9–12 are devoted to the *Equations dealt in Ch. 2*. In particular, Ch. 11 deals with the *Fermat Equation $x^n+y^n=z^n$* in rational integers $n>2, x>0, y>0, z>0$. Under various conditions, for example when $y-x$ is composed of fixed primes, it is shown that there are only finitely many solutions.

The style is rather leisurely. The authors have not rushed to give a proof of the most general result in a chapter; they first deal with particular cases so that the reader may take the strain of the proof gradually. However, in certain cases they have proved a generalization by a method different from the particular.

Each chapter is in three parts. The first part contains the statements of all the results to be subsequently proved. The second part contains the proofs of these results and the third part gives an account of the developments related to the previous results. Thus an account of the results of important topics which could not be included in the text is available in the Notes.

The book is highly recommended to the mathematicians (teachers, students and researchers) and to the readers who have to solve problems using diophantine equations.

*** * * Parallel Algorithms for Matrix Computations.** Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 1990, 197 pp., ISBN 0-89871-260-2.

This book describes a selection of important parallel algorithms for matrix computations. It reviews the current status and provides an overall perspective of parallel algorithms for solving problems arising in the major areas of numerical linear algebra, including: 1) *Direct Solutions of Dense, Structured or Sparse Linear Systems*; 2) *Dense or Structured Least Squares Computations*; 3) *Dense or Structured Eigenvalues and Singular Value Computations* and 4) *Rapid Elliptic Solvers*.

The book emphasizes computational primitives whose efficient execution on parallel and vector computers is essential to obtain high performance algorithms.

This volume consists of two comprehensive survey papers on important parallel algorithms for solving problems arising in the major areas of numerical linear algebra—direct solution of linear systems, least squares computations, eigenvalue and singular value computations, and rapid elliptic solvers, plus extensive up-to-date bibliography (2,000 items) on related research.

Contents: *Parallel Algorithm for Matrix Computations*, by K. A. Gallivan, R. J. Plemmons and A. H. Sameh (Reprinted from SIAM Review, March 1990, 82 pp.); *Parallel Algorithms for Sparse Linear Systems*, by M. T. Heath, E. G. Y. Ng and B. W. Peyton (Original, 42 pp.); *A Bibliography on Parallel and Vector Numerical Algorithms*, by J. M. Ortega, R. G. Voigt and C. H. Romine (Original, 73 pp.).

W. FULKS, Complex Variables, An Introduction. Marcel Dekker, Inc., New York, 1993, 402 pp., ISBN 0-8247-9079-0.

This book is designed for a one semester *Introduction to the Subject of Complex Variables*. The major way in which the exposition of this material differs from other texts on the same subject is in its organization, although it also has some unique simplifying approaches to certain topics.

The most unorthodox feature is the early introduction of the integral, particularly the integral in polar coordinates. This enables to prove preliminary forms of the Cauchy theorems for polynomial and rational functions. This in turn permits early access to the evaluation of some real integrals which are quite difficult by elementary calculus methods, and so provides an early demonstration of the computational utility of the subject.

A second important consequence of the early introduction of the integral is that it allows to define the complex logarithm by $\log z = \int_1^z d\varphi/\varphi$. From this definition the ambiguity

(multivaluedness) is discussed in an easy and geometrically natural way based on the paths integration from 1 to z .

Another unusual organizational feature of the discussion is the treatment of series before the main Cauchy theorems and Taylor's theorem. This leads to define the exponential and trigonometric functions by their power series. These functions are then seen as the natural extensions to the complex plane of the corresponding real functions. An aspect of the discussions that the author believes to be quite useful is the treatment of analytic continuation. The emphasis is on practical methods of producing continuations and includes many examples and exercises. They show the utility of continuation as a tool, for example, in the evaluation of integrals.

One pays considerable attention to applications, those centering on *Potential Theory*. There are discussions of *Fluid Flow*, of *Heat Conduction*, of *Conformal Mappings* and of *Evaluation of Integrals*. One introduces the *Laplace Transform* and shows its use in solving initial value problems for differential equations, including a Heaviside theorem. The last chapter is devoted to an *Elementary Systematic Introduction to Logarithmic Potential Theory*. This chapter contains a final section on *Fourier Series* and their use in some problems of partial differential equations.

In the development of the material in some sections one has occasionally relegated difficult proofs to the Exercises. These are frequently peripheral topics or are advanced topics not essential to an undergraduate course. But for those students and instructors who want to go more deeply into these subjects the exercises provide a rather complete outline with sufficient hints. A prime example is provided by the discussion of the relationship between the rectangular and the polar equations of a curve.

The exercises are classified in an approximate ordering of their difficulty, from straightforward applications of the textual development and conceptually deeper, to exercises intended to challenge the students.

C.W. BROWN, *Matrices and Vector Spaces*. Marcel Dekker, Inc., New York, 1991, 309 pp., ISBN 0-8247-8419-7.

The author designed *Matrices and Vector Spaces* to be used as a textbook for a one-semester or two-quarter course in linear algebra. The book is written for those students who have a serious interest in mathematics, and for those college instructors who want a challenging book for their students. This book can also be used by seniors and graduate students who find they need a more thorough grounding in the fundamentals of the subject. In this respect the titles of the chapters are very significant and transparent: 1. *Matrices and Linear Systems of Equations*; 2. *Vector Spaces*; 3. *Determinants*; 4. *Inner Product Spaces*.

The text in *Matrices and Vector Spaces* is deliberately formal in style. The idea is to present the subject matter in the same way most students will see it in advanced courses and applications. One has taken care to present clear and concise statements of all definitions and theorems in the book. In particular, whenever an argument is given that uses an algorithm or a reduction process, one has tried to present several steps in the procedure rather than resort to induction. In this way, one makes the algorithm or reduction process clear for future use.

Most of the material in *Matrices and Vector Spaces* is presented from a matrix point view when appropriate. The corresponding statements for linear transformations, if not given

in the text, can be found in several advanced text on the same topic. A list of references is provided for this purpose at the end of this book. All definitions, theorems, and important formulas are clearly displayed in the text with an accompanying number. Notation for specific ideas, for example $\mathbb{R}$ for the field of real numbers, is introduced in the main body of the text as a needed. A glossary of notation has been provided at the back of the book for the reader's convenience. Answers and suggestions for selected problems have also been included at the end of the text.

CLAUDE GASQUET et PATRIK WITOMSKI, *Analyse de Fourier et Applications. Filtrage, Calcul numérique, Ondelettes*. Masson Editeur, Paris, 1990, 368 pp.

Ce livre écrit par deux distingués professeurs de l'Université Joseph Fourier de Grenoble, est un ouvrage de mathématiques appliquées autour des thèmes *Analyse de Fourier, Filtrage et Traitement du Signal*. L'objectif est d'apporter au lecteur mathématicien un éclairage sur l'utilisation des notions fondamentales d'analyse qu'il apprend et au lecteur physicien un cadre théorique dans lequel les formules "bien connues" trouvent leur justification.

L'ouvrage est ainsi une illustration naturelle, au niveau des études d'ingénieur ou des licences, maîtrises et magistères de mathématiques ou de physique, d'une formation de base en analyse, calcul numérique et modélisation. Il veut éveiller l'intérêt des étudiants pour la cohérence entre les domaines suivants: la pratique de l'*Intégrale de Lebesgue* et des *Distributions*, l'*Analyse de Fourier*, le *Filtrage* et l'*Échantillonnage des Signaux*, l'*Analyse Temps-Fréquence* avec les *Transformées de Gabor* et en *Ondelettes*. Il permet ainsi une formation générale qui permettra ensuite d'aborder un travail plus spécialisé dans des directions diverses.

Pout faciliter la lecture et la compréhension, l'exposé est présenté sous forme de petites leçons favorisant une assimilation à la carte. Ces leçons sont regroupées en douze chapitres qui permettent d'évoluer rapidement à l'intérieur du livre.

Destiné plus particulièrement aux étudiants ci-dessus désignés, il sera également utile aux chercheurs ou aux ingénieurs qui s'intéressent aux questions de modélisation et de traitement des signaux.

YVES MEYER, *Wavelets: Algorithm & Applications* (transl. and revised by Robert D. R y a n). Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 1993, XI+ 133 pp., ISBN 0-89871-309-9.

Wavelet analysis, an exciting new theory on the forefront of scientific thought, is a unifying concept that interprets a large body of scientific research. For example the application of wavelet-based techniques to image compression has major economic implications. In the expanding field of signal and image processing, this book provides a clear set of concepts, methods, and algorithms adapted to a variety of nonstationary signals and numerical image processing problems.

Professor M e y e r, from the University Institute of France, one of the world's leading experts in wavelet research, presents with equal skill and clarity the mathematical background and the major wavelet applications, ranging from the digital telephone to galactic

structure and creation of the universe. Never before have the historic origins, the algorithms, and the applications of wavelets been discussed in such detail, providing a unifying presentation accessible to scientists and engineers across all disciplines and levels of training.

The book is based on a series of lectures presented at the Spanish Institute and is now being published for the first time in English. The c o n t e n t is: Ch. 1, *Signals and Wavelets*; Ch. 2, *Wavelets from a Historical Perspective*; Ch. 3, *Quadrature Mirror Filters*; Ch. 4, *Pyramid Algorithms for Numerical Image Processing*; Ch. 5, *Time-Frequency Analysis for Signal Processing*; Ch. 6, *Time-Frequency Algorithms Using Malvar's Wavelets*; Ch. 7, *Time-Frequency Analysis and Wavelet Packets*; Ch. 8, *Computer Vision and Human Vision*; Ch. 9, *Wavelets and Fractals*; Ch. 10, *Wavelets and Turbulence*; Ch. 11, *Wavelets and the Study of Distant Galaxies*; Index.

Written specifically for scientists with diverse backgrounds, the material is presented in a manner that will appeal to both experts and nonexperts alike. It is a valuable tool for engineers and scientists (from graduate students to experts) faced with signal and image processing problems. It is also the answer to anyone asking, "What are Wavelets?"

SABINE VAN HUFFEL and JOOS VANDEWALLE, **The Total Least Squares Problem: Computational Aspects and Analysis**. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 1991, 300 pp., ISBN 0-89871-275-0.

The *Total Least Squares* (TLS) represents a technique which synthesizes statistical and numerical methodologies for solving problems arising in many application areas.

This is the first book devoted entirely to total least squares. The authors, leaders in showing how to use TLS for solving a variety of problems, especially those arising in signal processing context, give a unified presentation of the TLS problem — a description of its *Basic Principle*, the *Various Algebraic, Statistical and Sensitivity Properties* of the problem are discussed, and generalizations are presented. Applications are surveyed to facilitate uses in an even wider range of applications. Whenever possible, comparison is made with the well-known least squares methods.

C o n t e n t s: *Introduction; Basic Principles of the Total Least Squares Problem; Extensions of the Basic Total Least Squares Problem; Direct Speed Improvement of the Total Least Squares Computations; Iterative Speed Improvement for Solving Slowly Varying Total Least Squares Problems; Algebraic Connections Between Total Least Squares and Least Squares Problems; Sensitivity Analysis of Total Least Squares Problems; Sensitivity Analysis of Total Least Squares and Least Squares Problems in the Presence of Errors in all Data; Statistical Properties of the Total Least Squares Problem; Algebraic Connections Between Total Least Squares Estimation and Classical Linear Regression in Multicollinearity Problems; Conclusions.*

A basic knowledge of numerical linear algebra, matrix computations and some notion of elementary statistics is required by the reader. Some background material is included to make reasonably self-contained this elegant and clear monograph.

CHARLES VAN LOAN, **Computational Frameworks for The Fast Fourier Transform**. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 1992,

X+ 273 pp., ISBN 0-89871-285-8.

This is the most comprehensive treatment of FFTs to date. It captures the interplay between mathematics and the design of effective numerical algorithms — a critical connection as more advanced machines become available. The author uses a stylized Matlab notation which will be familiar to those engaged in high-performance computing.

The Fast Fourier Transform (FFT) family of Algorithms has revolutionized many areas of scientific computation. The FFT is one of the most widely used algorithms in science and engineering, with applications in almost every discipline.

C o n t e n t s: Ch. 1, *Radix-2 Frameworks. Matrix Notation and Algorithms; The FFT Idea; The Cooley-Tukey Factorization; Weight and Butterfly Computations; Bit Reversal and Transportation; The Cooley-Tukey Framework; The Stockham Autosort Frameworks; The Pease Framework; Decimation in Frequency and Inverse FFTs;* Ch. 2, *General Radix Frameworks; The General Radix Ideas; Index Reversal and Transportation; Mixed-Radix Factorizations; Radix-4 and Radix 8 Frameworks; The Split-Radix Framework;* Ch. 3, *High Performance Frameworks; The Multiple DFT Problem; Matrix Transposition; Large Single-Vector FFT Problem; The Multi-Dimensional FFT Problem; Distributed Memory FFTs; Shared Reversal Memory FFTs;* Ch. 4, *Selected Topics; Prime Factor FFTs; Convolution; FFTs of Real Data; Cosine and Sine Transforms; Fast Poisson Solvers; Bibliography; Index.*

This book is essential for professionals interested in linear algebra as well as those working with numerical methods. The FFT is also a great vehicle for teaching key aspects of scientific computing.

J. CRONIN, Differential Equations; Introduction and Qualitative Theory (2nd Ed., Revised and Expanded). Marcel Dekker, Inc., New York, 1994, 370 pp., ISBN 0-8247-9189-4.

The main purpose of the changes in this edition is to make the book more effective as a textbook. To this end, the following steps have been taken.

First, the material in Ch. 1 has been rearranged and augmented to make it more readable. Motivations for *Studying Existence Theorems and Extension Theorems* are given. The exercises at the end of the chapter have been extended to illustrate the theory more extensively, and the examples from chemistry and biology introduced are treated in detail.

In Ch. 2, a treatment of the *Sturm-Liouville Theory* has been added and treated in outline form with the purpose to make clear the relation between initial value problems and boundary value problems, to describe how the Sturm-Liouville theory is rooted in elementary linear algebra, and to show that the Sturm-Liouville problem and its solution form a concrete example in functional analysis.

The text in all the chapters has been reviewed and in some parts clarified and improved, for instance the proof of the *Instability Theorem* has been substantially simplified by allowing use of *Lyapunov Theory*.

Finally, a solution manual is available for instructors. The solutions manual (of more than 100 pages) includes detailed solutions for almost all the exercises at the ends of the chapters and also a detailed analysis of the examples given at the end of Ch. 1: the *Volterra Population Equations*, the *Hodgkin-Huxley Nerve Conduction Equations*, the *Field-Noyes Model of the Belousov-Zhabotinsky Reaction*, and the *Goodwin Model of Cell Activity*. The

analysis given for these examples shows the student the importance of the existence and extension theorems and also provides an early introduction to the use of *Geometric Methods*. As the theory is developed in later chapters, it is applied to these examples in the exercises at the ends of the corresponding chapters. Thus fairly extensive analyses of these examples are given in the solutions manual.

The solutions in the manual are written in detail so that the instructor can easily judge the length of an outside assignment and so that the students can read them.

A.G. LEONOV und V. REITMANN, *Attraktoreneingrenzung für nichtlineare Systeme*. Teubner-Texte zur Mathematik, Bd. 97, Teubner Verlag, Leipzig, 1987, 196 S., ISBN 3-322-00427-9.

Im Buch werden verschiedene *Klassen nichtlinearer Systeme, darunter solche mit chaotischem Lösungsverhalten*, mit der direkten Methode von Ljapunow, der Tschaplygin-Methode, der nichtlinearen Reduktionsmethode, der Kegeltheorie und anderen Hilfsmitteln untersucht.

Dabei gelingt es, Stabilitätsaussagen und Attraktorinklusionen zu gewinnen. Die betrachteten Systeme stammen aus der *Hydromechanik* (Lorenz-System), der *Chemie* (Rössler-System), der *Physik* (MASER-System), der *Theorie der automatischen Steuerung* (Digitalsysteme der Phasensynchronisation) und anderen Gebieten.

O. ARIBO, E.D. AXELROD and M. KIMEL (Eds.), *Mathematical Population Dynamics*. Proceedings of the 2nd International Conference, May 17-20, 1989, Rutgers University, New Brunswick, N. J.; Marcel Dekker, Inc., New York, 1991, 785 pp., ISBN 0-8247-8424-3.

This volume appears at a time when there are *Important Developments Occurring in Mathematics and Biology*. From the mathematical viewpoint, there has been recent progress in models of structured populations. There are models that identify variables such as age, size, amounts of specific constituents, or other characteristics of individuals (genes, cells, or organisms) to reproduce the time evolution of heterogeneity within populations. On the most abstract level, these new models require analysis of integral, functional, and partial differential equations (frequently achieved by the analysis of semigroups of operators) replacing the traditional ordinary differential equation formalism. There also has been recent progress in stochastic models involving a discrete or continuous variety of types of individuals within a population.

The selection of subjects was motivated by recent developments in the mathematical theory of population and by new data available from various branches of molecular and cellular biology and biomedical sciences. The topics included have been selected from those presented at the Second International Conference on Mathematical Population Dynamics. No attempt has been made to cover all of the traditional topics in population biology. The chapters are grouped within the following topics: *Structured Populations; Ordinary and Partial Differential Equations Models; AIDS and Theory of Epidemics; Stochastic Models; Cell Cycle Kinetics; Proliferation and Tumor Growth and Genetics and Molecular Biology*.

This volume is intended for mathematicians, statisticians, biologists, and medical

researchers who are interested in recent advances in analyzing changes in populations of genes, cells and tumors; in the natural history of cancer; and in epidemiological topics such as AIDS.

T.H. BANKS, H.R. FABIANO and K. ITO (Eds.), **Identification and Control in Systems Governed by Partial Differential Equations**. Proc. of the *Conference on Control and Identification of Partial Differential Equations*, Mount Holyoke College, South Hadley, Mass., July 11-16, 1992; Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 1992, 234 pp.

The objectives of this conference were to provide up-to-date information on recent developments and results in *Control, Identification, and Mathematical Modeling of Partial Differential Equations* and to stimulate further research in the area of control and identification for distributed parameter systems. This volume, which provides a record of some of the presentations and discussions that took place during this meeting, reflects current trends in control and system identification for partial differential equations. In the following we summarize the contributions contained in this volume, grouped according to research topics.

Mathematical Modeling. *Mathematical Models for Piezoceramic Actuators in Smart Material Structures* are presented by H.T. Banks and R.C. Smith. The models are developed for actuation via bending moments and in plane forces induced by imbedded piezoceramic patches. F.T. Tran, J.F. Scroggs and K.J. Bachmann consider *Mathematical Models for Flow Dynamics in a Vertical Reactor for High Pressure Vapor Transport of Materials for Compound Semiconductors*. Numerical simulations are performed to address design issues for the Scholz geometry of the reactor.

Parameter Estimation. G. Crosta considers the *Problem of Estimating Spatially Varying Conductivity in the One Dimensional Heat Equation*. Injectivity and stability of the inverse of parameter-to solution mapping are studied under various Cauchy data on the parameter. A *Statistical Analysis of Error Distribution in the Least Squares Parameter Estimation Problem* is studied by G. B. Fitzpatrick and G. Yin. A bootstrap method for inference, including confidence intervals and tests of normality, is developed. M. Kroller and K. Kunisch develop a software based on MATLAB for the *Estimation of Parameters in Two Point Boundary Value Problems*. This can be used to investigate how different discretizations and regularization terms affect the behaviour of solutions to an ill-posed inverse problem. The *Problem of Estimating Flexural Rigidity in a von Kármán Plate Equation* using dynamical point-observations of deformations is discussed by L. White. The problem is formulated as a norm-constrained minimization problem in Hilbert spaces.

Optimal Control. A. Bensoussan and F. Bernhard consider *Robust Stabilization of Infinite Dimensional Systems*. The problem is formulated as a standard problem of H_∞ -optimal control. The optimal feedback synthesis based on Riccati equations is derived. A *Shape Optimization Problem* that arises in *Design of a Forebody Simulator* for a free-jet engine test facility is studied by J. Borggárd, J. Burns, E. Cliff and M. Gunzberger. Sensitivity equations for the state equation with respect to the design parameters are derived and used for accurate gradient calculations. A *Linear Quadratic Regulator Problem for Systems Governed by Second Order Elliptic Equations* with Neumann

boundary control is studied by L. J i and G. C h e n. The cost functional is defined by the square of norm of tracking error at the sensed points on the boundary. J. A. R e n e k e considers a *Control Algorithm Based on the Reproducing Kernel Hilbert Space Method for Hereditary Systems* where the system is described by the input-output covariance function. Numerical examples are given to illustrate the approach.

Feedback Stabilization. The *Problem of Global Exponential Stabilization of a von Kármán Plate* by boundary velocity feedback is discussed by M.E. B r a d l e y and I. L a s i e c k a. The energy multiplier method and microlocal analysis are used to obtain global exponential stability of the controlled dynamics. C.I. B y r n e s, D.S. G i l l i a m and V.I. S h u b o v consider a *Boundary Control for Burgers' Equation with Flux Control at one End*. It is shown that the uncontrolled dynamics are not asymptotically stable. *Well-Posedness of Feedback Control Systems Described by the Triple (A, B, C) on Hilbert Spaces* is discussed by K.A. M o r r i s. It is shown that well-posed systems remain stable under bounded feedback control. H. O z b a y and J. T u r i consider a *Class of Control Systems Described by Singular Integro-differential Equations*. Existence of a finite dimensional stabilizing compensator (output feedback control) is shown employing modern frequency domain techniques. *Feedback Control of a One-link Flexible Robot Arm* is discussed by T. J. T a r n, A.K. B e j c z y and C. G u o. The Problem is formulated as a nonlinear distributed parameter control problem.

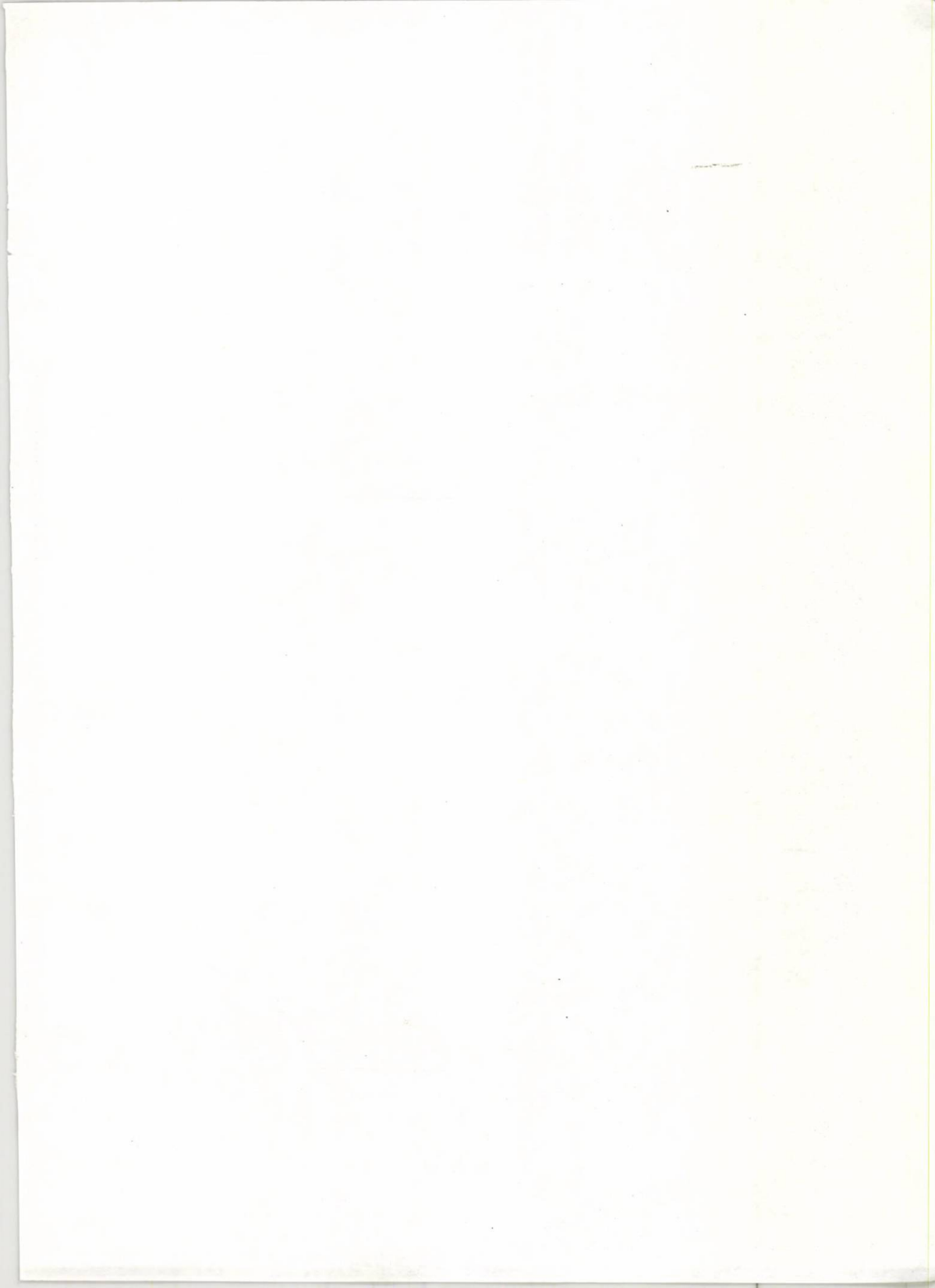
K. BINDER and D.W. HEERMANN, *Monte Carlo Simulation in Statistical Physics. An Introduction*. Springer Series in Solid-State Sciences, Vol. 80, Springer Verlag, 1988, VIII+127 pp. 34 figs., ISBN 3-540-19107-0.

This book is the product of a series of lectures given by the authors at the Summer School held at Figueira da Foz (Portugal) in 1987. Its intended scope is to lead the students to a thorough understanding of the *Basic Concepts of Monte Carlo Simulation* and to provide a practical guide on how to work with this simulation method.

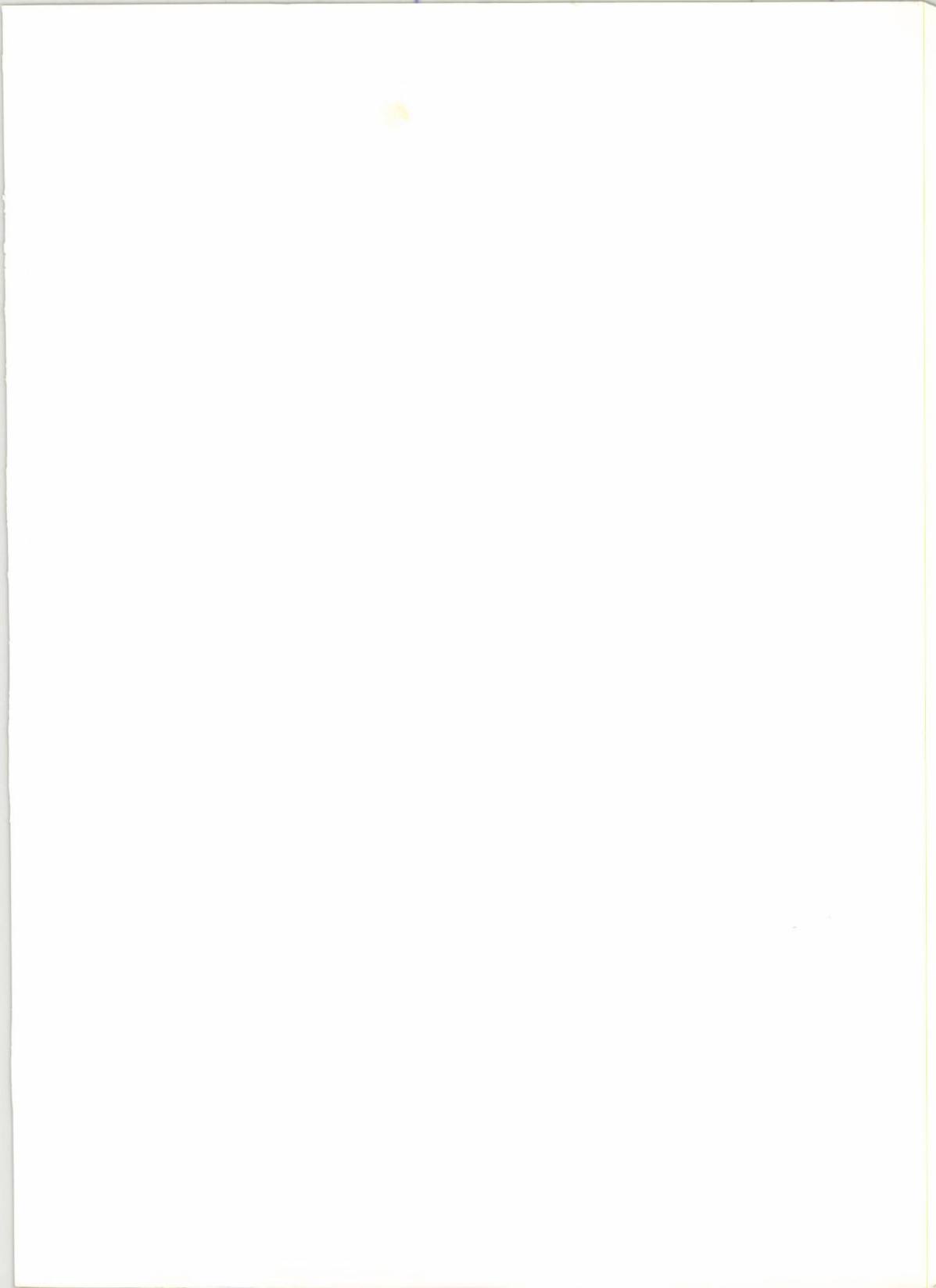
The book is organized into two parts. Part 1 contains an introduction to *The Theoretical Foundations of the Monte Carlo Method*. Applications of this method in Statistical Physics are also presented. Part 2 contains a guide to *Practical Work with the Monte Carlo Method*. Many simple worked examples are included to make the book a stimulating teaching text. The book also contains two appendix, a references list and a subject index.

The book will be of great value to graduate students and researchers interested in Monte Carlo Method.

Viorel Stancu







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